Traffic Management under Boundedly Rational User Equilibrium: Day-to-day Toll Strategy

Dawn Spruijtenburg
University of Twente
sdspruijtenburg@gmail.com

ABSTRACT

We consider the static traffic assignment model in which travelers are boundedly rational in their route choice. This assignment introduces uncertainty, since generally multiple Boundedly Rational User Equilibrium (BRUE) solutions exist. In this paper, we propose a day-to-day toll strategy that steers the network from an observed BRUE to a desired BRUE. We prove that the stationary state of this toll strategy is the desired flow, and we show by example that the strategy can achieve any flow.

Keywords

Traffic management, Boundedly Rational User Equilibrium, day-to-day process, toll

SRC 2016, November 30, 2016, The Netherlands

INTRODUCTION

Strategic decision making for traffic authorities is often based on static traffic assignment models. When measures are implemented, intended to change the distribution, authorities often seek for measures that improve the observed traffic distribution towards an assignment with less traffic congestion, a fairer assignment, or an assignment with less total travel time or pollution [1]. In this paper, we propose a strategy that can be used as a model by authorities to improve traffic situations.

When researching traffic networks, often the notion of the User Equilibrium (UE) is used to describe the stationary state of traffic networks, and it is assumed that the UE is observed in real life. The UE is a traffic assignment in which no traveler can improve his or her travel time by unilaterally changing routes [2]. This notion of UE to model networks, is based on naive decision-making assumptions. It is assumed that travelers are perfectly rational in their route choice: they make selfish decisions, are perfectly informed, and can perfectly assess the effects of the decisions they make [3].

Bounded Rationality was introduced in the context of traffic problems by Mahmassani and Chang in the form of the Boundedly Rational User Equilibrium (BRUE) [4]. Empirical evidence suggests that the BRUE is more appropriate to describe the real-world equilibrium than the UE. The BRUE is the result of a traffic assignment when we consider travelers to be boundedly rational. The BRUE does not assume perfect rationality as the User Equilibrium does, because travelers do not always choose the shortest path [5]. Instead of wanting the best possible route, travelers are now assumed to be content with their (suboptimal) travel times when they are 'acceptable'. In contrast to the UE, where under basic assumptions only one solution can be found, there is a range of solutions for the BRUE [4]. Note that, even though multiple 'stable' solutions exist, only one solution is observed in real life. This difference makes the UE a lot more convenient to work with, mathematically.

The fact that there are multiple solutions to the BRUE leads to difficulties for an authority, since it is unknown which BRUE arises in practice after a policy intervention [8]. However, the fact that there are multiple solutions to the BRUE can also be seen as an advantage, which we use in our strategy.

In this paper, we design a method that steers the network from a given assignment to a desired, better BRUE. From an equity perspective, it would be best to move to the UE (the UE is a special case of the BRUE). From a system perspective, we would like to move to a solution that lowers the total costs. In this paper, we work towards a desired BRUE assignment that minimizes travel costs, assuming we can find this assignment efficiently. Note that the desired can be any assignment, and therefore other objectives can be pursued as well, for example minimizing pollution.

In fact, the strategy we propose is a day-to-day toll strategy. Every day a different toll is collected. Where traditional approaches typically collect tolls on a link level (see, e.g., Guo), advances in mobile applications allow for a more personalized toll system like proposed here [6]. As mentioned, we aim for a BRUE solution as the desired assignment, since it allows us to remove the toll, and the assignment will stay. This is due to the fact that every BRUE is a stable solution, because travelers are satisfied with their (suboptimal) routes in a BRUE. Therefore, tolls only have to be collected for a limited amount of time, which, we hypothesize, is preferable over collecting tolls continuously

PROBLEM FORMULATION

Traffic Assignment

The traffic assignment is done given a fixed demand. $G = (V, E)$ is the directed traffic network, with V the set of nodes and E the set of links $e = (i, j)$ with $i, j \in V$. Given is a set of origin-destination pairs (OD-pairs), $K \subseteq V \times V$, for which the static demand $d_k > 0, k \in K$. One OD-pair is referred to as OD-pair $k \in K$. An OD-pair k is connected by the set P_k of simple directed paths (routes). The set of all paths is the union of the paths that connect each OD-pair, $P = \cup_{k\in K} P_k$. A feasible traffic flow for fixed demand $d \in \mathbb{R}^{|K|}$ is (f, x) such that

$$(f, x) \in \mathcal{F} := \{(f, x) \in \mathbb{R}^{|P|} \times \mathbb{R}^{|E|} \,|\, \Lambda f = d, \Delta f = x, f \geq 0\}.$$

Then, f is the vector which describes the flow on each route, while x is the vector that describes how the flows, described by f, are distributed over the links.

The OD-path incidence matrix $\Lambda \in \mathbb{R}^{|K|\times|P|}$ is defined such that $\Lambda_{kp} = 1$ if $p \in P_k$, so if path p connects OD-pair k, and $\Lambda_{kp} = 0$ otherwise. The link-path incidence matrix is denoted by and defined as $\Delta \in \mathbb{R}^{|E|\times|P|}$ where $\Delta_{ep} = 1$ if edge e is on path p, and $\Delta_{ep} = 0$ otherwise.

For every link $e \in E$ a flow-dependent travel time (latency, cost) $l_e(x)$ is defined. We assume that link costs are separable, continuous, convex and strictly monotonically increasing. Separable link costs ensure that the travel time of a link does not depend on the traffic located on other links, i.e., $l_e(x) = l_e(x_e)$. The cost of a path is defined as the sum of all links on that path, $c_p(f) = \sum_{e\in p} l_e(x_e)\ \forall p \in P$. The cost of a path depends on the link flows indirectly, as the route flows f will determine the link flows x.

Boundedly Rational User Equilibrium

The BRUE is a traffic distribution in which travelers do not necessarily take the shortest path, but a path of which the costs are ‘acceptable’. We introduce an indifference band, which is the maximum difference in travel cost between the shortest path and the chosen path that is still perceived acceptable. Formally, given an indifference band $\varepsilon \in \mathbb{R}_+^{|K|}$, traffic flow $(f, x) \in \mathcal{F}$ with cost vector $c(f)$ is called a Boundedly Rational User Equilibrium (BRUE), if $\forall k \in K$ and $\forall p \in P_k$ the following condition is satisfied:

$$f_p > 0 \Rightarrow c_p(f) \leq \min_{q\in P_k} c_q(f) + \varepsilon_k. \tag{1}$$

A BRUE flow is a feasible flow $(f, x) \in \mathcal{F}$ that satisfies (1), a flow in which every path that carries flow has a cost within the specified range. We assume that ε is constant over time. The condition described in (1) was first discussed by Mahmassani and Chang, after which it was formalized by, among others, Di et al., and Lou et al. [4] [7] [8].

We define a penalty function as introduced by Ye and Yang [9]:

$$\gamma_p(f) \coloneqq \max\{\tilde{c}_p(x,\tau) - \varepsilon_k, \tilde{\mu}_k\}, p \in P_k, k \in K, \tag{2}$$

This penalty function is equal to the costs of the shortest path when a path satisfies the BRUE condition, otherwise it is equal to the amount by which the path is ‘too expensive’. In (2), $\tilde{c}_p(x,\tau) \coloneqq c_p(x) + \tau_p, p \in P_k, k \in K$, is the experienced travel costs, that consists of the travel time and the induced toll $\tau \in \mathbb{R}^{|P|}$, and $\tilde{\mu}_k \coloneqq \min_{q\in P_k} \tilde{c}_q(x), k \in K$, is the shortest path including toll.

We define the total penalty as $f^T\gamma(f)$, we multiply the penalty with the flow that encounters that penalty. When $f^T\gamma(f) = f^T(\Lambda^{\mathrm{T}}\tilde{\mu}_k)$, then f is a BRUE assignment as in (1), because then each penalty is equal to the cost of the shortest path, and therefore each path satisfies the BRUE condition. Indeed, it directly follows that $f_p > 0 \Rightarrow c_p(x) - \varepsilon_k \leq \mu_k$, which equals condition (1).

Discrete Adjustment Process

For defining the strategy, we consider a Discrete Adjustment Process (DAP) to describe travelers' behavior with respect to route choice from day-to-day. We follow the formulation of Guo et al. [6]. The process describes how a new assignment at time epoch $n + 1$, $f(n + 1)$, is achieved by adding part of the current assignment at n, $f(n)$, to part of a new assignment, $g(n)$.

The DAP of route flows is formally formulated as:

$$f(n+1) = \big(1 - \lambda(n)\big)f(n) + \lambda(n)g(n),\ n = 1,2,\ldots \tag{3}$$

The flow $f(n + 1)$ on day $n + 1$ then consists of two parts, i.e., $f(n)$, the travelers that did not change their route, and $g(n)$, the travelers that did change their route. The adjustment ratio $\lambda(n)$ shows that only part of the travelers reconsiders their choice each day, the other part stays on their route. The adjustment ratio is $\lambda(n) \in (0,1]$ and $g(n)$ satisfies a condition which is described in (8).

Travelers reconsider their choice when they are on a path that they consider as ‘too expensive’. This means that that path has unacceptable travel cost, i.e., $c_p(x) > \min_{q\in P_k} c_q(f) + \varepsilon_k$. They will then switch to a path that they perceive (based on the costs of day n) as less costly. Additionally, $\lambda(n)$ has a mathematical purpose, it determines whether the process described in (3) converges to a stationary solution. A stationary solution is a solution where the assignment at time epoch n is the same assignment as at time epoch $n + 1$, i.e. $f(n) = f(n + 1)$. In other words, a stationary solution is an assignment in which no traveler wants to change routes.

DESIRED ASSIGNMENT TOLL STRATEGY

In this section, we propose a day-to-day toll strategy that steers the network from an observed BRUE to a desired assignment. The strategy presumes that a desired assignment has been defined and calculated, for example the BRUE with minimal travel times, or minimal pollution.

Notations

We introduce a reformulation so that we can describe all BRUE solutions. This is formulation is equivalent to the formulation used by Eikenbroek et al. and Di et al. [10] [7]. Parameter $\tilde{\rho} \in \mathbb{R}^{|P|}$, $0 \leq \tilde{\rho}_p \leq \varepsilon_k, p \in P_k, k \in K$, is defined as follows for $p \in P_k, k \in K$:

$$\tilde{\rho}_p \coloneqq \begin{cases} \mu_k + \varepsilon_k - c_p(f) & \text{if } c_p(f) \leq \mu_k + \varepsilon_k \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We define a minimization problem corresponding to our notation:

$$\min \tilde{z}\,(\tilde{\rho}, f, x) = \sum_{e\in E} \int_0^{x_e} l_e(\omega)d\omega + \tilde{\rho}^T f$$
$$\text{s.t. } (f, x) \in \mathcal{F}. \tag{5}$$

Note that this notation corresponds to the Beckmann formulation when $\tilde{\rho} = 0$ [11], which is the formulation used to find the regular UE.

The formulation in (5) finds a BRUE assignment for given $\tilde{\rho}$. For each fixed $\tilde{\rho}$, optimization problem (5) has a unique solution with respect to the link flows. That means that each $\tilde{\rho}$ determines a BRUE assignment. Further, Di et al. proved that any BRUE assignment can be described by a unique $\tilde{\rho}$ [7].

We consider the system of Karush-Kuhn-Tucker (KKT) optimality conditions that correspond to optimization problem (5) [12]. These conditions are both necessary and sufficient conditions for optimality, as $\tilde{z}$ is convex in (f, x) for given $\tilde{\rho}$. We introduce Lagrange multiplier vector $(\beta, \pi, \delta) \in \mathbb{R}^{|E|}\times\mathbb{R}^{|K|}\times\mathbb{R}^{|P|}$.

Proposition 1. Any $(f, x) \in \mathcal{F}$ is a global optimal solution of optimization problem (5), if and only if (f, x) satisfies the following system with $(\beta, \pi, \delta) \in \mathbb{R}^{|E|} \times \mathbb{R}^{|K|} \times \mathbb{R}^{|P|}$ with $\delta \geq 0$:

$$l(x) - \beta = 0, \qquad \delta^T \tilde{z} = 0,$$
$$\tilde{\rho} + \beta^T \Delta + \Lambda^T \pi - \delta = 0, \;\; (f, x) \in \mathcal{F}.$$

We substitute $\beta_e = l_e(x_e)$ for all $e \in E$ and from the complementarity condition $\delta^T \tilde{z} = 0$ we find:

$$f_p > 0 \Rightarrow c_p(f) + \tilde{\rho}_p + \pi_k = 0, \forall p \in P_k, k \in K. \tag{6}$$

Proposition 1 describes a condition that must be fulfilled for a solution to be optimal, thus for a solution to be a BRUE solution described by $\tilde{\rho}$. We use Proposition 1 later in Theorem 1 to prove that the stationary solution of the toll strategy is the desired assignment.

Toll Strategy

In the previous paragraph, we showed that we can describe any assignment by a $\tilde{\rho}$. Now say that $(f^d, x^d) \in \mathcal{F}$, (f^d, x^d) satisfies condition (1), is the desired assignment, described by $\tilde{\rho}^d$, the assignment that we want to achieve by applying the toll strategy. The toll on route p for day $n+1$ is defined as:

$$\tau_p(n+1) := \begin{cases} \tilde{\rho}^d{}_p & \text{if } c_p(f(n)) + \tilde{\rho}^d{}_p = \min\left\{c_p(f(n)) + \tilde{\rho}^d{}_q\right\}, \\ \varepsilon_k + \tilde{\rho}^d{}_p & \text{otherwise.} \end{cases} \tag{7}$$

Intuitively, the toll ensures that every path that is 'too expensive', the path does not satisfy the BRUE condition in (1) for the new assignment, gets a high toll. The toll makes the path even more expensive and therefore serves as incentive for changing routes. Then each user bases their route choice on day $n+1$ on the travel time of day n and the toll on day $n+1$, which induces the penalty: $\tilde{\gamma}_p(f) = \max\left\{c_p(f(n)) + \tilde{\rho}^d{}_p, c_q(f(n)) + \tilde{\rho}^d{}_q\right\}, p \in P_k, k \in K$, in which q is the shortest path described by $\tilde{\mu}_k$. The penalty for a path $p \in P$ becomes: $\tilde{\gamma}_p(n) = c_p(f(n)) + \tilde{\rho}^d{}_p$. Then, every traveler experiences a penalty equal to their travel cost plus the $\tilde{\rho}^d$ value of their current path.

We use (3) to update the flows. The adjustment ratio is again $\lambda(n) \in (0,1]$ and $g(n)$ satisfies:

$$g(n) \begin{cases} \in \Phi(n), & \text{if } \Phi(n) \neq \emptyset, \\ = f(n), & \text{if } \Phi(n) = \emptyset. \end{cases} \tag{8}$$

The set $\Phi(n)$ is defined as:

$$\Phi(n) = \left\{g \in \mathcal{F}_f \middle| g^T\left(c(f(n)) + \tilde{\rho}^d\right) < f(n)^T\left(c(f(n)) + \tilde{\rho}^d\right)\right\}.$$

The set $\Phi(n)$ thus contains all assignments in which travelers changed to paths with lower penalty values, and one of these assignments is chosen to update the flows.

Theorem 1 can be explained intuitively as follows. In a stationary flow pattern, no user wants to change their route, they are satisfied with their travel time including toll. Theorem 1 shows that this stationary assignment is then the desired assignment described by $\tilde{\rho}^d$.

Theorem 1. If $f(n)$ is a stationary flow pattern of the system defined in (3) with the toll strategy defined in (7), i.e. $f(n) = f(n+1)$, then $f(n)$ is a flow that solves optimization problem (5) with parameter $\tilde{\rho}^d$.

Proof. Suppose that the flow is stationary, then $f(n) = f(n+1)$, thus $g(n) = f(n)$, i.e. the set $\Phi(n)$ must be empty, so it is true that:

$$\left(g - f(n)\right)^T\left(c(f(n)) + \tilde{\rho}^d\right) \geq 0, \forall g \in \mathcal{F}_f.$$

Then, $f(n)$ gives a solution to the following Linear Program:

$$\min_y y^T\left(c(f(n)) + \tilde{\rho}^d{}_p\right) \text{ s.t. } \Lambda y = d, y \geq 0. \tag{9}$$

We consider the system of Karush-Kuhn-Tucker (KKT) necessary and sufficient optimality conditions that correspond to this system. We introduce Lagrange multiplier vector $(\pi, \delta) \in \mathbb{R}^{|K|} \times \mathbb{R}^{|P|}$. Any $y \in \mathcal{F}_f$ is a global optimal solution of the Linear Program described in (9), if and only if y satisfies the following system with $\delta \geq 0$: $c(f(n)) + \tilde{\rho}^d{}_p + \Lambda^T \pi - \delta = 0, \quad \delta^T y = 0.$

From the complementarity condition $\delta^T y = 0$ we find that: $y_p > 0 \Rightarrow c_p(f(n)) + \tilde{\rho}^d{}_p + \pi_k = 0, \forall p \in P_k, k \in K.$

As this is the same condition as in (6), we can conclude that the stationary assignment is indeed the desired assignment, and because we chose $\tilde{\rho}$ to be a BRUE assignment, we have achieved a BRUE solution. ■

Therefore, we can conclude that the stationary solution of the toll strategy is the desired assignment, and if we were to achieve this stationary solution, the strategy is usable. As the stationary solution is a BRUE solution, we can remove the tolls and the desired assignment will remain.

Illustration

In the previous section, we showed that the stationary solution of the toll strategy is the desired solution, independent of the starting point. In this section, we show, by numerical example, that we achieve this stationary solution, $\tilde{\rho}^d$, when starting from several initial (non-BRUE) assignments. For the simulation, we use the Braess network [13], as in Figure 1. Here, we consider only one OD-pair, with origin 1 and destination 4, and paths $P_1 = \{(1,2,4), (1,3,4), (1,2,3,4)\}$. The demand will be $d_1 = 6$. We show how the toll strategy works when we start from a non-BRUE assignment. Note that the strategy also works if we start from a BRUE assignment.

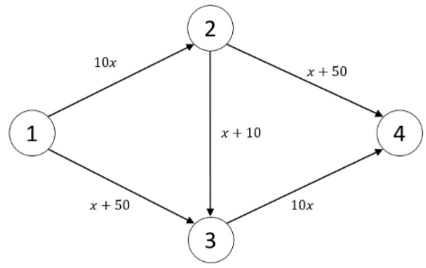

Figure 1: Braess Network used for Simulation

We show the toll strategy. The assignment we approach using the toll strategy described in (7) is an assignment that could for example be the assignment with minimal emission, for which $\tilde{\rho}^d = (15, 9.5, 2.5)$. The corresponding flow pattern is $f^d = (1, 1.5, 3.5)$. In Figure 2 we start from non-BRUE assignments and let the system evolve according to the toll strategy in (7), using the Network Tatonnement Process (NTP) as described by Ye and Yang to update the route flows [9]. The NTP is a method which (uniquely) describes how travelers choose to change their routes.

Furthermore, $\lambda(n) = 0.1$ in each step, $\beta = 0.5$ where β is a parameter as used in the NTP. For this toll process, the value of $\varepsilon = 15$ is used. The updating of flows is stopped when the total penalty as in (2) does not change by more than 0.01 in one step anymore. The process does not stop when we reach a BRUE, but only when we achieve the stationary, and therefore desired assignment.

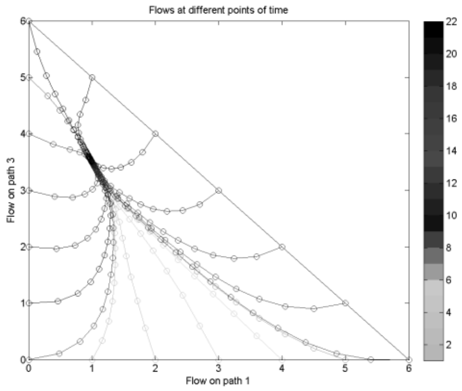

Figure 2: Toll Strategy: Path flows over time

Figure 2 shows the path flows over time. Each shade represents a different starting flow, numbered 1 to 21, each circle represents a flow at a certain time for that starting flow. The location of the circle shows the flow on path 1 and path 3 at that time. The flow on path 2 can be determined as well, since the total demand is known. Figure 2 shows that all flows terminate close to the point $f^d = (1, 1.5, 3.5)$, which is the desired assignment.

Figure 2 shows the evolution when the toll strategy as described in (7) is applied. It can be seen that, independent of the starting point, the desired assignment is achieved. The assignments reached are stationary, meaning they will not change over time anymore. This is due to the fact that the final assignments are BRUE assignments, and travelers are satisfied with their routes in BRUE assignments. As the desired assignment is a BRUE, we end up in a BRUE solution from each starting point.

CONCLUSION

In this paper, a toll strategy for improving traffic assignments is proposed. We prove that when the flow does not evolve, the desired assignment is achieved. We show that in the Braess network, and thus in general, this strategy can achieve the desired assignment. Further research proves the convergence to the desired assignment.

ROLE OF THE STUDENT

Dawn Spruijtenburg is an undergraduate student who worked under the supervision of Oskar Eikenbroek for this project. The topic was proposed by the supervisor. Other experts involved were Eric van Berkum and Fokko Jan Dijksterhuis. The student worked out the proofs, the simulation and the writing of the paper herself.

REFERENCES

[1] M. Abdulaal and L. J. LeBlanc, "Continuous Equilibrium Network Design Models," *Transportation Research Part B: Methodological,* vol. 13, no. 1, pp. 19-32, 1979.

[2] J. G. Wardrop and J. Whitehead, "Correspondence. Some Theoretical Aspects of Road Traffic Research," *Proceedings of the Institution of Civil Engineers,* vol. 1, no. 5, pp. 767-768, 1952.

[3] J. Conlisk, "Why Bounded Rationality?," *Journal of Economic Literature,* vol. 34, no. 2, pp. 669-700, 1996.

[4] H. S. Mahmassani and G.-L. Chang, "On Boundedy Rational User Equilibrium in Transportation Systems," *Transportation Science,* vol. 21, no. 2, pp. 89-99, 1987.

[5] S. Zhu and D. Levinson, "Do people use the shortest path? An empirical test of Wardrop's first principle," *PloS one,* vol. 10, no. 8, 2015.

[6] R.-Y. Guo, H. Yang and H.-J. Huang, "A Discrete Rational Adjustment Process of Link Flows in Traffic Networks," *Transportation Research Part C: Emerging Technologies,* vol. 34, pp. 121-137, 2013.

[7] X. Di, H. X. Liu, J.-S. Pang and X. J. Ban, "Boundedly Rational User Equilibria (BRUE): Mathematical Formulation and Solution Sets," *Procedia-Social and Behavioral Sciences,* vol. 80, pp. 231-248, 2013.

[8] Y. Lou, Y. Yin and S. Lawphongpanich, "Robust Congestion Pricing under Boundedly Rational User Equilibrium," *Transportation Research Part B: Methodological,* vol. 44, no. 1, pp. 15-28, 2010.

[9] H. Ye and H. Yang, "Rational Behavior Adjustment Process with Boundedly Rational User Equilibrium," *Transportation Science,* 2017.

[10] O. A. Eikenbroek, G. J. Still, E. C. van Berkum and W. Kern, "The Boundedly Rational User Equilibrium: A Parametric Analysis with Application to the Network Design Problem," 2017.

[11] M. Beckmann, C. McGuire and C. Winston, "Studies in the Economics of Transportation," Yale University Press, New Haven, 1956.

[12] U. Faigle, W. Kern and G. Still, "Algorithmic Principles of Mathematical Programming," *Springer Science & Business Media,* vol. 24, 2013.

[13] D. Braess, "Über ein Paradoxon aus der Verkehrsplanung," *Mathematical Methods of Operations Research,* vol. 12, no. 1, pp. 258-268, 1968.