Hearing the Shape of a Trapezoid Drum

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ABSTRACT
In this article we take a small step toward answering the famous question "Can one hear the shape of a drum?" Recently, Grieser and Maronna [1] have shown that if the drum is triangular, the answer is yes, you can.
We investigate this question for trapezoid drums. We define a subset of trapezia and prove that for drums of this shape the answer to the question is also positive. To do this, it turns out that we need to investigate periodic orbits in a triangle. In particular, we prove a lower limit on the length of families of periodic orbits.

Keywords
Inverse spectral problem, periodic orbits

INTRODUCTION
Suppose we want to make some truly new music, how can we do this? Perhaps if we use drums that are not circular, but shaped like a triangle, maybe a trapezium or even wilder shapes, how would those drums sound?
Similar to how the length of a guitar string determines which tones it can play, the shape of a drum determines what tones it can produce. If we have a drum of shape Ω, physics tells us that these tones can be written as a countably infinite list of frequencies $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ called the spectrum.
Given Ω, computing the λ_k is called the forward spectral problem and is in general quite hard. We are interested in the inverse spectral problem: given the numbers λ_k, what properties of Ω can we compute? In a famous lecture from 1966 Mark Kac [2] popularized this problem as "Can one hear the shape of a drum?"
Unfortunately, it turns out the answer is no, not all properties of Ω are completely determined by the spectrum. Webb, Gordon and Wolpert [3] have constructed two different shapes whose spectra are identical.
However, in 2012 it was discovered that if it is a priori known that the shape Ω is a triangle, all properties of this triangle can be recovered from the spectrum [1]. In this text, we will prove that this is also the case for acute enough trapezia.
The central idea of our proof is to use a theorem by van den Berg and Srisatkunarajah [4] that states the area and perimeter of a shape can be computed from the spectrum, together with a theorem due to Hillairet [5] which states that, under certain conditions, a family of periodic orbits leaves a characteristic fingerprint in the spectrum. The data from this fingerprint, together with area and perimeter, will be sufficient to determine all properties of a trapezium.

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It turns out that the crucial step in the proof that enables our use of Hillairet's theorem is a classification of families of periodic billiard ball orbits in triangles, and, in particular, a lower bound on their length. Finding this bound will be the main subject of this article.
We will first introduce billiard ball orbits by the important example of the Fagnano triangle. Our next step is to sort all orbits into five groups, based on how they travel around the triangle. We then show for each of these groups that such families are longer than our lower bound. We conclude by using this lower bound to prove that all properties of an acute enough trapezium can be recovered from the spectrum, so that one can indeed hear the shape of an acute enough trapezium.

BILLIARD BALL ORBITS
A cornerstone of our discussion is the idea of billiard ball orbits corresponding to a shape Ω. These orbits are the paths an idealized billiard ball would travel on a billiard table with shape Ω. Idealized means that the ball is a point particle, experiences no drag and bounces elastically off the edge of the domain per the standard reflection rules.
Of particular interest to us are the orbits that eventually return to their original situation, these are called periodic orbits. Such orbits will trace out the same path forever.
As an example, consider a square billiard table with sides of length L, shooting a billiard ball perpendicular to a side will cause it to travel along a periodic orbit with length $2L$ as it will start bouncing back and forth between the sides.
We have also mentioned families of orbits. A periodic orbit is part of a family if the starting position can be moved slightly and this different starting position also creates a periodic orbit with the same period. The set of all orbits obtained by these slight displacements is the family.

The Fagnano triangle
Before we start with our investigation into families of periodic orbits, we provide an example of a periodic orbit called the Fagnano triangle, which is the triangle formed by the bases of the altitudes from all three angles. This example will introduce some important techniques in a fairly simple setting.
Lemma 1. *Suppose ABC is an acute triangle. The shortest periodic orbit is given by the Fagnano triangle. The length of this orbit is $2h_i \sin(\alpha_i)$ where α_i is any angle of the triangle and h_i the corresponding altitude.*
Proof: Our first point of order should be to prove that the Fagnano triangle is indeed a periodic orbit. This is a geometric proof that is not relevant for our later discussion so we refer to [6] for a proof.
To show that the Fagnano triangle is the shortest periodic orbit we start with an arbitrary orbit and apply a series of steps that make the path shorter, we show that this results in the Fagnano triangle.
Suppose we have a periodic billiard ball orbit. It necessarily hits each side at least once, so let a be a point where the orbit hits the side BC, b such a point on AC and c on AB. Connecting a, b and c by straight lines as in Figure 1 provides a path that is certainly shorter than the

original orbit, but does not necessarily obey the reflection laws at a, b and c.
Now reflect the triangle ABC and the point c across sides AC and BC to obtain Figure 1. The path $abca$ now has the same length as the path $c'bac''$. We can obtain a shorter path by considering the straight line between c' and c'' instead, so instead of the current path consider the path where a and b are on the line $c'c''$.

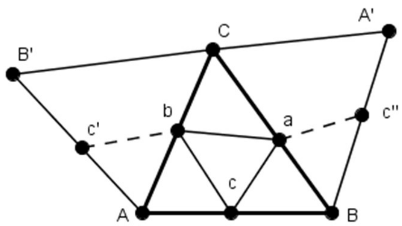

Figure 1: *The unfolding of the trajectory* abc. *The triangle* ABC *and point* c *are reflected across* AC *and* BC, *the dashed lines between* c', b *and* a, c'' *are the reflections of the line-segments* bc *and* ca *respectively.*

Using the opposite angle theorem this guarantees that in fact the trajectory does satisfy the reflection laws at a and b, when we consider the corresponding path in the original triangle.
By the above process we have reduced finding the shortest closed path connecting the three sides of the triangle, to finding the shortest line between AB' and BA' such that $|B'c'| = |Bc''|$.
Either by explicit calculation or a symmetry argument this minimum is unique and corresponds to the unique choice of $|B'c'|$ such that $\angle Ac'b = \angle Bc''a$.
Translating this result back to our path in ABC, we find that the shortest possible path does in fact satisfy the reflection laws at c as well. Hence the shortest closed trajectory connecting all three sides is given by a periodic billiard ball orbit.
To see that this orbit is in fact the Fagnano triangle, note that we know that the Fagnano triangle is such an orbit, hence it unfolds to a straight line satisfying $\angle Ac'b = \angle Bc''a$ in Figure 1. The line satisfying this condition was unique, therefore the line that became our shortest orbit is in fact the Fagnano triangle.
Finding the length of the Fagnano triangle is a lengthy exercise in trigonometry which is not enlightening for our purposes, so it is omitted. □
The important technique in this proof is the strategy of unfolding the orbit to a straight line, which we did by reflecting the triangle across the sides that were hit by the orbit. We will use this idea to first classify the possible orbits and then prove a lower bound on their length.

Characterizing families of orbits

We will now try to characterize all possibilities for families of orbits in a triangle, using the ideas from unfolding. We will categorize the families by their reflection type, which is a list of which sides they hit in what order.
Naming the sides of the triangle a, b, c a type could for example be $abcb$, for an orbit that starts on side a, then hits b, followed by c and b before returning to the starting position on side a.
For a sequence of a's, b's and c's to be a valid type we need to make sure that all three letters appear at least once, since no periodic orbit is possible using only two sides of a triangle. Furthermore, at no point may consecutive letters be the same, since an orbit can never hit the same side twice in a row. Note that this also means that the first and last character must be different.
Not all types that look different at first glance really are, so there are certain equivalences we want to mention. Two types are equivalent if

- one can be obtained from the other by cyclic translation. As an example, we consider $abcbac$ the same as $bcbaca$ since this is the same type with a different side chosen as starting position.
- one is the other reversed, for example $abcbab$ and $babcba$ since this is just the same orbit traversed in the other direction.
- one can be obtained by a permutation of the characters. For example, the types $abcbcac$ and $cababcb$, which are equivalent by replacing a by c, b by a and c by b. This corresponds to renaming the sides of the triangle.

This concept of types is convenient for our unfolding technique, since it describes the sequence of reflections we need to make to unfold the orbit to a straight line, as follows.
Suppose the type is $x_0 x_1 \dots x_n$ where each x_i is either a, b or c. In order to unfold this type of orbit to a straight line, we start with a triangle T and name the sides a, b and c. First we reflect T across side x_1 and call this new triangle T_1. Reflect T_1 across side x_2 and repeat this, reflecting T_i across side x_{i+1} until we arrive at T_n. The orbit is now a straight line between a point p on side x_0 of T and a point q on side x_0 of T_n that intersects each x_i in order.
To guarantee periodicity the points p and q must have the same distance to corresponding vertices of the triangle and the lines must make the same angle with their respective sides to guarantee the reflection law. Note that the reflection law, together with the opposite angle theorem, guarantees that an orbit unfolds to a straight line.
As an example, the type of the Fagnano triangle is abc or equivalently cba, taking the left most triangle in Figure 1 as our initial triangle, the shape obtained there is exactly the unfolding of type cba.
We will always use a to refer to side BC, b for AC and c for side AB when considering a triangle ABC.
Now we are ready to state our characterization result.
Lemma 2. *All types of orbits either*

1. *have an odd number of reflections.*
2. *contain a type equivalent to* $abcabc$.
3. *are equivalent to* $abcbac$.
4. *are equivalent to* $[ab][cb]$.
5. *are equivalent to* $[ab]c[ac]b$.

Here $[x_0 \dots x_n]$ *means an arbitrary number of repetitions, possibly zero, of the characters* $x_0 \dots x_n$.
Proof of Lemma 2: Suppose our type has an odd number of reflections, then we are done since this is case 1.
Now suppose it has an even number of reflections. If one side only appears once we can, by permuting the symbols, choose this to be side a and the first side it hits after a can be chosen to be b. By cyclic translation the type can then be made to start with ab. The remainder of the type must be filled with b's and c's only, so it must be $ab[cb$, since we cannot use two identical characters consecutively and the total number of reflections is even. Hence if one side is used only once, the type is of category 4.
The proof continues along the same lines but is convoluted and not very enlightening. □

Lower bound on the length of families

For orbits with types corresponding to cases 1, 2 and 4 we will now present our proofs that, in an acute triangle, either they do not correspond to a family of orbits, or the corresponding family is longer than $2h$ if h is the altitude from an angle larger than $\frac{\pi}{3}$. The other cases can be proven similarly and provide no additional insight. All 5 cases combined provide a lower bound of $2h$ for the length of families in acute triangles.

Lemma 3. *A type with an odd number of reflections never corresponds to a family of orbits.*

Proof: We will give the idea of the proof here, for a formal elaboration see [5, p. 112].

The central concept is that if we travel along the path of a billiard ball, whenever we reflect off the edge of the table our direction changes. So if we are accompanied by a billiard ball slightly to our left, after a reflection it will be to our right. At every reflection, the other orbit will change from our left to our right or vice versa.

Hence, if after an odd number of reflections we return to our initial position, the other billiard ball cannot have returned. This means the other ball's orbit cannot be periodic. Therefore, an orbit with an odd number of reflections cannot be part of a family. □

We remark here that traversing such an orbit twice might make it part of a family, since then the total number of reflections becomes even. This is ok since the types $abcb$ and $abcbabcb$ are considered different.

Lemma 4. *Any family of orbits whose type contains a type equivalent to* $abcabc$ *is longer than* $2h$ *if* h *is the altitude from an angle larger than* $\frac{\pi}{3}$.

Proof: This proof is very similar to the proof we gave for the Fagnano triangle, and in fact closely related to the Fagnano triangle.

First we apply all necessary transformations to make sure $..a..b..c..a..b..c..$ appears in the type. Let $p_1, \dots, p_6$ be the points at which the orbit hits the sides corresponding to the a, b, c, a, b, c and instead of the original orbit, consider the closed path connecting these six points in order, analogous to what we did in the proof of Lemma 1. The unfolding corresponding to the type of this path, $abcabc$, is shown in Figure 2.

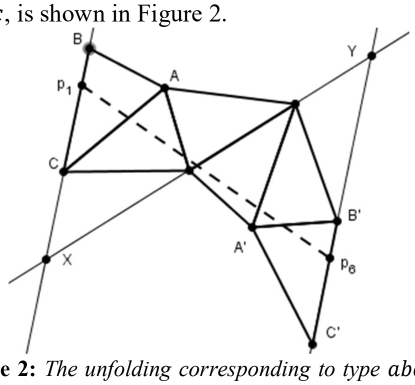

Figure 2: *The unfolding corresponding to type* $abcabc$. *The dashed line corresponds to a periodic orbit if* $|Bp_1| = |B'p_6|$. *Note that the lines through* BC *and* $B'C'$ *are always parallel.*

As in the proof of Lemma 1, consider the straight line connecting p_1 and p_6 instead of the current path, this is shorter than the trajectory and thus shorter than the original orbit.

Investigating the unfolding a little closer, we note that the sides BC and $B'C'$ are parallel. This can be proven by computing the angles at X and Y in terms of the angles of the triangle and using the Z-angle theorem.

This means that all lines between these sides that satisfy the periodicity condition, $|Bp_1| = |B'p_6|$, are parallel and have the same length.

A specific example of such a line is the unfolding of the Fagnano triangle traversed twice. Because it is periodic it satisfies the periodicity condition, the type of this orbit is $abcabc$ thus this unfolding transforms it into a straight line between BC and $B'C'$. According to Lemma 1, the length of this doubled orbit is $4h_i \sin(\alpha_i)$. If α_i is greater than $\frac{\pi}{3}$ we have that the $\sin(\alpha_i) > \frac{1}{2}$, so the length of the double Fagnano orbit, and thus the original orbit, is more than $2h_i$. □

The next lemma deals with a different situation, where the type is not completely known. This makes the unfolding more interesting.

Lemma 5. *Any family of orbits whose type is equivalent to* $[ab][cb]$ *has length greater than* $2h$ *if* h *is the altitude from an angle greater than* $\frac{\pi}{3}$.

Proof: Suppose ab is repeated n times and cb is repeated m times. We will actually consider the equivalent type $b[ab]c[bc]$, where ab is still repeated n times and bc is repeated $m-1$ times, because this provides a more useful unfolding. Note that both m and n are at least 1.

The unfolding is shown in Figure 3 for $n = 1, m = 2$. Any orbit of this type unfolds to a straight line between points p and q on sides AC and $A'C'$, such that $|Cp| = |C'q|$. Displacing the starting position corresponds to moving the points p and q to and from C and C'.

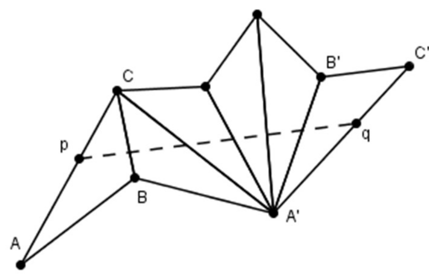

Figure 3: *The unfolding corresponding to* $b[ab]c[bc]$ *for one repetition of both blocks, if* $[bc]$ *is repeated more often extra wedges are added around* A'. *The dashed line is member of a family of orbits only if* AC *and* $A'C'$ *are parallel and* $|Cp| = |C'q|$.

If the distance $|pq|$, the length of the family, is to remain constant when varying p and q we need that AC and $A'C'$ are parallel. Using α for the angle corresponding to A, β for the angle of B and γ for C this condition becomes $2n\gamma = 2m\alpha$ by the Z-angle theorem with $A'C$. Furthermore, we need that $2n\gamma < \pi$ and $2m\alpha < \pi$ to allow a line between AC and $A'C'$ to intersect the line-segment CA'.

We also know that ABC must be acute, so $\alpha + \gamma > \frac{\pi}{2}$ since $\beta < \frac{\pi}{2}$. From this we can conclude that $\frac{\pi}{2} < \alpha + \gamma < \frac{\pi}{2n} + \frac{\pi}{2m}$, therefore $1 < \frac{1}{n} + \frac{1}{m}$. But then n or m must equal 1. Without loss of generality we may assume that $n = 1$, otherwise we start with the equivalent type where a and c are swapped, as this swaps the values of n and m.

This allows us to compute the length of this family very easily. Because AC and $A'C'$ are parallel, all lines satisfying the periodicity condition between these sides are the same length. The choice $p = A$, $q = A'$ satisfies this condition, and the length of this line is twice the altitude from A since $n = 1$.

If h is the altitude from A we are done, since then the length of the family is $2h$. If h is the altitude from B or C there is some more work to be done, but this is an exercise in trigonometry which we will skip here. □

HEARING THE SHAPE OF A TRAPEZIUM

We are now finally ready to apply these results to our original question “Can one hear the shape of a drum?” We will answer this question for acute enough trapezia. These are trapezia where the acute angles are on the same side and their sum is less than $\frac{2\pi}{3}$, as in Figure 4.

Theorem 1. *Given the spectrum of an acute enough trapezium, all properties of the trapezium can be recovered.*

Throughout this section we will use the naming conventions established by Figure 4. As a convention, we assume $x \geq y$, otherwise we mirror the trapezium to swap x and y.

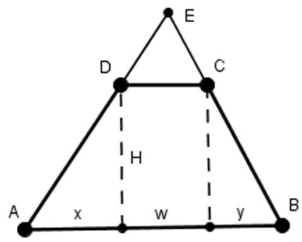

Figure 4: *An acute enough trapezium $ABCD$, together with the triangle completion ABE. The two dashed lines are the height of the trapezium; they are also the outermost orbits in the family of periodic orbits travelling between the base AB and top CD.*

The proof of Theorem 1 requires the following two Theorems; we will not go into detail on their proofs here as they are not suited for this text.

Theorem 2. *From the spectrum of Ω one can compute the area A and the perimeter P.*

Proof: See [4]

Theorem 3. *If there is a unique family of periodic orbits of a certain length, the length of this family and the area within Ω covered by the family can be computed from the spectrum.*

Proof: See [5, p. 135].

We are now ready to prove Theorem 1.

Proof of Theorem 1: Suppose we have the spectrum of an acute enough trapezium. First we use Theorem 2 to find the area A and perimeter P of the trapezium.

Every acute enough trapezium has a family of orbits reflecting straight up and down between the base and the top. If we can show that this is the only family of this length, Theorem 3 will provide us with the length of this family, which is $2H$, and the area covered by the family, which is H multiplied by the width of the top w.

To show that this family is the only one of length $2H$ we first consider families of orbits that only hit the base and the slanted sides. These families will also be families of the triangle ABE.

If the triangle completion is acute we can invoke our lower bound on the length of families in an acute triangle. Note that the angle at E must be greater than $\frac{\pi}{3}$, because the base angles of the trapezium add up to less than $\frac{2\pi}{3}$. Therefore, any family of orbits in this triangle must be longer than twice the altitude from E, so certainly longer than $2H$.

If the new angle is not acute we can invoke a classical result that states that the shortest orbit in an obtuse triangle is the height from the obtuse angle [7]. Hence also in this case any family of orbits in the triangle completion must be longer than $2H$.

Any other family of orbits in the trapezium must at some point reflect off the top, and by acuteness must also hit the base. Hence such a family is also necessarily longer than $2H$, and strictly longer if it differs from the family reflecting straight up and down.

Therefore, the family of orbits reflecting straight up and down between the base and top is the only family of length $2H$, allowing us to invoke Theorem 3 and find the values of H and w.

It is a straightforward exercise to find A and P in terms of x, y, H and w. Under the assumption $x \geq y$ there is a unique solution for x and y to this pair of equations. This shows that the spectrum determines x, y, H and w, which together completely determine the trapezium. □

CONCLUSION

We first investigated the families of orbits that can occur in a triangle and found a lower bound for their length in terms of the altitude from an angle larger than $\frac{\pi}{3}$.

This characterization informed our definition of acute enough trapezia, such that we could guarantee the family of height-orbits in an acute enough trapezium was the unique family of that length. This in turn allowed us to invoke a result by Hillairet which, together with a theorem by van den Berg and Srisatkunarajah, provided enough data to fully determine the trapezium.

Thus, we have shown that for an acute enough trapezium all properties can be recovered from the spectrum, provided we a priori knew the spectrum corresponded to such a trapezium. This mirrors the result by Grieser and Maronna for triangles.

ROLE OF THE STUDENT

Luuk Verhoeven was an undergraduate student working under the supervision of Walter van Suijlekom when research for this article was performed. The work presented in this text was done as part of the student’s bachelor thesis and the Radboud-FNWI Honours programme. The topic was proposed by the supervisor, the general idea of the proof arose during discussions while the details were established by the student. The writing was done by the student.

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