On curves with constant curvatures

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ABSTRACT

One of the fields of research of Computer Aided Geometric Design is approximating complex curves by simpler curves. Curves with constant curvatures are useful tools for these purposes. However, parametrizations of such curves are not always easily given. In this paper we will derive several necessary and sufficient geometric conditions for a curve to have constant curvatures, both in Euclidean geometry and in affine geometry.

Keywords

Euclidean geometry, affine geometry, differential geometry, curvature, curve

INTRODUCTION

Many people have an intuitive idea of the curvedness of a curve. A straight line is not curved at all, while a constantly curved curve in the plane is a piece of a circle. In three-dimensional space we add helices as curves with constant curvatures. These are all examples from Euclidean geometry, but more possibilities are provided by looking at higher-dimensional space or "a different geometry".

Since Klein's Erlanger Programm [5] in 1872, geometry is regarded as the study of invariants under a certain transitive transformation group. In Euclidean geometry this is the group of rigid transformations, consisting of translations, rotations and reflections. The natural invariant of this group is the distance between pairs of points. Affine geometry studies geometric invariants under the larger group of volume preserving linear transformations, also known as equi-affine transformations [1]. Every equi-affine transformation can be

represented as $T(x) = Ax + b$, where the fact that T is volume preserving means that $\det A = 1$. The natural invariant of this group is (signed) volume.

As Euclidean geometry speaks of *Euclidean* curvatures, affine geometry speaks of *affine* curvatures. In this paper we will look at curves with constant curvatures for both types of curvature in m-dimensional space.

EUCLIDEAN DIFFERENTIAL GEOMETRY

Euclidean arc length

In this section we will give an analytical description of Euclidean arc length and curvatures. For a regular differentiable curve $\gamma : I \rightarrow \mathbb{R}^m$ the arc length of $\gamma([a, b])$ can be approximated by subdividing the interval $[a, b]$ and calculating the length of the resulting inscribed polygonal curve [2, 10]. By refining the subdivision sufficiently we obtain the arc length

$$\int_a^b |\gamma'(t)| dt, \tag{1}$$

where $| \cdot |$ denotes the Euclidean norm. We say that γ is parametrized by arc length if its tangent vector has unit length at every point. Then the length of $\gamma([a, b])$ is given by $b - a$. We will write γ as function of s if it is parametrized by arc length. Any derivatives are then with respect to s.
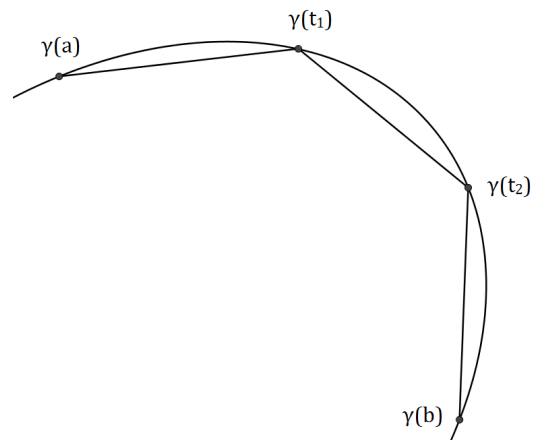

Figure 1: Geometric definition of arc length

Euclidean curvatures

In the well known Frenet-Serret formulas the curvature and torsion are obtained by constructing an orthonormal frame $\{t, n, b\}$. A similar construction can be made for curves in $\mathbb{R}^m$. If $\gamma'(s), \dots, \gamma^{(m)}(s)$ are linearly independent we can use the Gram-Schmidt Orthogonalization Process to obtain an orthonormal frame $\{F_1(s), \dots, F_m(s)\}$, see also [3]. From this moving frame we can express the curvatures $\kappa_1(s), \dots, \kappa_{m-1}(s)$ as follows:

$$\begin{aligned} F_1' &= \kappa_1 F_2, \\ F_i' &= -\kappa_{i-1} F_{i-1} + \kappa_i F_{i+1}, \\ F_m' &= -\kappa_{m-1} F_{m-1}, \end{aligned} \quad (2)$$

for $1 < i < m$. If $\gamma'(s), \dots, \gamma^{(m)}(s)$ are not linearly independent, we can find a curve in a lower-dimensional subspace with linearly independent derivatives that differs from γ only by a rigid transformation, so we will not consider this case.

AFFINE DIFFERENTIAL GEOMETRY

Affine arc length

To define arc length in affine geometry we cannot use the approach as in Euclidean geometry, since the length of line segments is not necessarily preserved under equi-affine transformations. However, instead of adding the lengths of the edges of an inscribed polygon, we can use the volume of simplices formed by points on the curve. As a result we obtain an expression for the affine arc length of $\gamma([a, b])$:

$$\int_a^b ||\gamma'(t), \gamma''(t), \dots, \gamma^{(m)}(t)||^{\frac{2}{m(m+1)}} \, dt, \quad (3)$$

where $||v_1, \dots, v_m||$ denotes the determinant of the matrix formed by the vectors $v_1, \dots, v_m$.
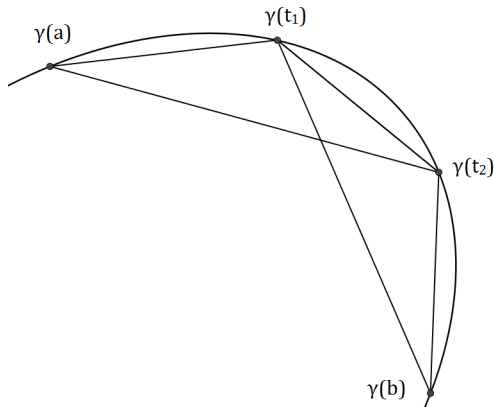
Figure 2: Geometric definition of affine arc length

Note that the definition of affine arc length depends on m, whereas the definition of Euclidean arc length does not. Furthermore, affine arc length is only defined if the curve has nonzero Euclidean curvatures. We say that γ is parametrized by affine arc length if

$$||\gamma'(t), \gamma''(t), \dots, \gamma^{(m)}(t)|| = 1 \quad (4)$$

for all $t \in I$. We will write γ as function of r if it is parametrized by affine arc length. Any derivatives are then with respect to r.

Affine curvatures

By differentiating equation (4) we see that $\gamma^{(m+1)}(r)$ is linearly dependent on $\gamma'(r), \dots, \gamma^{(m-1)}(r)$. Hence, there exist functions $k_1(r), \dots, k_{m-1}(r)$, called the affine curvatures, such that:

$$\gamma^{(m+1)} = k_1 \gamma' + \dots + k_{m-1} \gamma^{(m-1)}. \quad (5)$$

Explicit expressions for the k_i are given by

$$k_i = ||\gamma', \dots, \gamma^{(i-1)}, \gamma^{(m+1)}, \gamma^{(i+1)}, \dots, \gamma^{(m)}||. \quad (6)$$

This should be interpreted as follows: start with the determinant of the matrix with the first till m-th derivatives, then replace the i-th derivative by the $(m+1)$-th derivative.

IMPORTANCE OF CURVATURES

The curvatures, Euclidean or affine, are not only a way to measure properties of a given curve, but reversely certain curvatures determine a curve completely up to a transformation from the relevant transformation group. In Euclidean geometry of $\mathbb{R}^3$ this is known as the Fundamental Theorem of the Local Theory of Curves [2, 19], but the proof can be easily generalized to $\mathbb{R}^m$ and affine geometry. Further note that the curvatures are invariant under their respective transformations, hence they are well-defined.
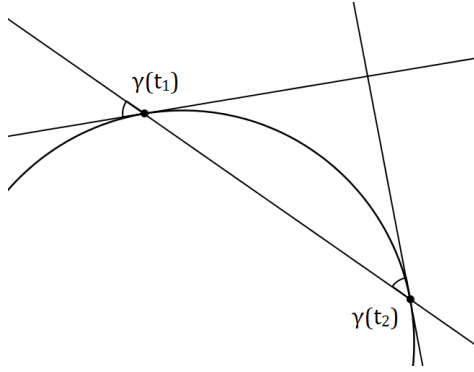
CONSTANT EUCLIDEAN CURVATURES

In this subsection we will derive two necessary and sufficient conditions for a curve to have constant Euclidean curvatures. One of these conditions is that the curve is equi-angular:

Definition 1. *A regular curve $\gamma : I \to \mathbb{R}^m$ is called equi-angular if the tangents at any two of its points make the same angle with the line segment connecting these points.*

We also need the following lemma which shows that if the first $i - 1$ curvatures are constant, then there is a very simple expression for the derivative of the i-th curvature.

Lemma 2. *Let $\gamma : I \to \mathbb{R}^m$ be parametrized by arc length with constant curvatures $\kappa_1, \dots, \kappa_{i-1}$ for $i < m$. Then*

$$\kappa_1^2 \kappa_2^2 \dots \kappa_{i-1}^2 \kappa_i \kappa_i' = \langle \gamma^{(i+1)}, \gamma^{(i+2)} \rangle. \quad (7)$$

Figure 3: Equi-angularity means that the designated angles are equal.

The two theorems are then given by:

Theorem 3. *A connected regular curve in $\mathbb{R}^m$ has constant Euclidean curvatures if and only if every arc of the curve is equi-angular.*

Theorem 4. *A connected regular curve in $\mathbb{R}^m$ has constant Euclidean curvatures if and only if the distance between two points on the curve does not depend on the actual positions of these points, but only on the arc length of the curve segment between the points.*
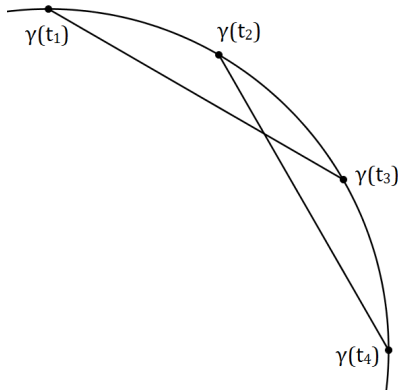
Figure 4: Theorem 4 means that the designated line segments have equal length if the corresponding arc lengths are equal.

Because the curvatures determine a curve completely, there exist certain normal forms for constant curvature curves. Using these normal forms a straightforward calculation shows that constant curvatures imply that the distance depends only on arc length.

The distance between $\gamma(t_1)$ and $\gamma(t_2)$ being a function of $t_2 - t_1$ can be rewritten as

$$\langle \gamma(t_1) - \gamma(t_2), \gamma'(t_1) - \gamma'(t_2) \rangle = 0, \qquad (8)$$

which is equivalent to equi-angularity.

Lastly, by differentiating equation (8) sufficiently many times we obtain

$$\langle \gamma^{(i+1)}, \gamma^{(i+2)} \rangle = 0 \qquad (9)$$

for any $1 \le i < m$. Combined with Lemma 2 and an induction argument we see that all curvatures are constant, proving both theorems.

CONSTANT AFFINE CURVATURES

Characterization in terms of volume

Analogous to the Euclidean case we will now derive two necessary and sufficient conditions for a curve to have constant affine curvatures. The first of these mirrors Theorem 4 perfectly:

Theorem 5. *A connected regular curve in $\mathbb{R}^m$ with nonzero Euclidean curvatures has constant affine curvatures if and only if the volume of the simplex formed by $m + 1$ points on the curve does not depend on the actual positions of these points, but only on the affine arc length of the curve segments between the points.*
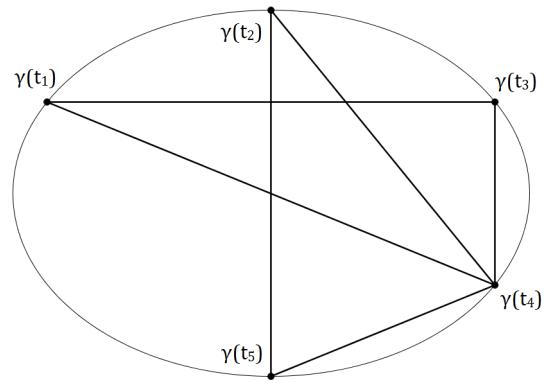
Figure 5: Above condition means that the areas of the triangles are equal if the corresponding pairs of affine arc lengths are equal.

The 'only if' part can be proved by taking two sets of $m+1$ points on the curve with equal affine arc length between corresponding points. Then there exists an equi-affine transformation mapping one simplex to the other, which implies that their volumes are equal.

Reversily, to prove the 'if' part we see that the volume of the simplex formed by $\gamma(t_1), \ldots, \gamma(t_{m+1})$ is given by

$$V = C \|\gamma(t_1) - \gamma(t_{m+1}), \ldots, \gamma(t_m) - \gamma(t_{m+1})\|, \tag{10}$$

for some constant C, depending only on m. If the volume is only a function of the differences $t_i - t_j$, then

$$\sum_{i=1}^{m+1} \frac{\partial}{\partial t_i} V = 0. \qquad (11)$$

By using the following algorithm for some arbitrary fixed $1 \le j \le m - 1$ we obtain $k'_j = 0$:

- Differentiate (11) once with respect to t_1, twice with respect to t_2, and so on until m times with respect to t_m, but skip t_j.

- Differentiate $m + 1$ times with respect to t_j.
- Set $t_1 = \ldots = t_m$.

Characterization in terms of offset curves

The second condition makes use of tangential offset curves:

Definition 6. *Let the regular curve γ in $\mathbb{R}^m$ be parametrized by affine arc length. The λ-tangential offset curve of γ is given by $\beta(r) = \gamma(r) + \lambda\gamma'(r)$.*
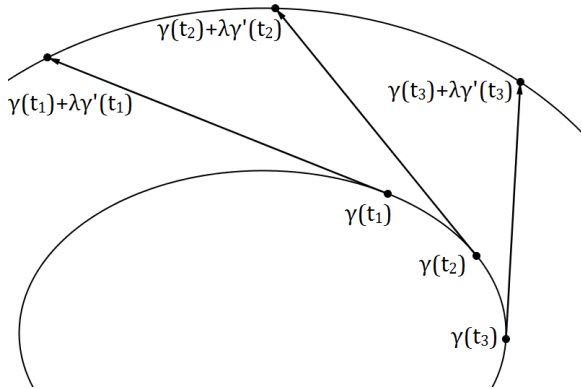
Figure 6: Geometric definition of a tangential offset curve

The theorem is then given by:

Theorem 7. *A regular curve in $\mathbb{R}^m$ nonzero Euclidean curvatures has constant affine curvatures if and only if the affine arc length of any λ-offset curve in the affine tangential direction is proportional to the affine arc length of the curve itself.*

The theorem follows immediately from the observation that we can rewrite the affine arc length of the λ-tangential offset curve of γ as

$$\int_a^b \left(1 + \sum_{j=1}^{m-1} (-\lambda)^{m-j+1} k_j(t)\right)^{\frac{2}{m(m+1)}} dt. \quad (12)$$

Note that the 'if' part follows by picking $\lambda_1, \ldots, \lambda_{m-1}$ nonzero and distinct and solving for k_j.

CONCLUSION

The most important results of this thesis are Theorems 4 and 5. Theorem 4 shows that a curve in $\mathbb{R}^m$ has constant Euclidean curvatures if and only if the distance between two points on the curve does not depend on the actual positions of the points, but only on the arc length of the curve segment between the points. Theorem 5 shows that a curve in $\mathbb{R}^m$ has constant affine curvatures if and only if the volume of the simplex formed by $m+1$ points on the curve does not depend on the actual positions of these points, but only on the affine arc length of the curve segments between the points.

It would be interesting to see if other Klein geometries,

like inversive geometry or projective geometry, follow the same rule. Generally, every Klein geometry can be seen as a k-point geometry, meaning that every k'-point invariant with $k' > k$ is a function of invariants of at most k points [4, 144]. If there exists a natural arc length parameter and natural curvatures, one could look if in this geometry a curve has constant natural curvatures if and only if a k-point invariant on the curve depends only on the natural arc length of the curve segments between the points. This will be part of future research.

ROLE OF THE STUDENT

Matthijs Ebbens was an undergraduate student conducting this research as bachelor project under supervision of prof. dr. G. Vegter. Having derived Lemma 2 and Theorem 3, the supervisor proposed this subject, wondering if there existed a similar theorem for affine curvatures. During half a semester Matthijs Ebbens derived the other theorems, proved them and wrote a report of which this paper is a shorter version. For the full proofs the reader is encouraged to contact the author. Two presentations were given about this work: once during a bachelor colloquium as part of the bachelor project, once as part of a symposium for all students who were writing their thesis at that time. The second supervisor prof. dr. H.W. Broer evaluated the thesis together with G. Vegter. Since finishing the thesis the student started to work on the open question posed in the conclusion, leading to the included proof of Theorem 5 that is simpler than the original proof.

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