On the growth efficiency of pebble-accreting planetesimals at 1 AU orbital distance from the star

Rico G. Visser
Anton Pannekoek Institute for Astronomy
University of Amsterdam
Supervisor: Dr. C.W.Ormel

ABSTRACT
One method to build planets is through pebble accretion, where a planetesimal sweeps up pebbles in the protoplanetary disk surrounding the star. I investigated the growth efficiency of planetesimals by numerically integrating the equation of motion for variable sized pebbles. These pebbles experience gas drag and interact gravitationally with the planetesimal and the star. The results are obtained by quantifying accretion in terms of a growth timescale. I obtained a finite transition between the flow-dominated regime [2] and the settling regime [7]. A barrier is preventing growth of planetesimals sweeping up particles smaller than 1 centimeter.

Author Keywords
Planet formation, gas dynamics, protoplanetary disk, planetesimal, fluid dynamics

INTRODUCTION
Planet formation is still a topic of interest since it is a process that is not understood. There are two main planet formation models: the disk-instability model and core accretion. In the first model clumps of gas in the disk collapse under gravity and form planets. One problem with this model is inconsistency in expectation and observation of the metallicity ratio of gas giants [5]. For the latter, small pebbles in the gas grow by sticking together and accumulating further until gravity takes over. One major problem with this model is that collisions for meter sized objects are not sticking anymore but rather destructive, which stagnates the growth process.[8]. I investigate an alternative model known as pebble accretion [3]. In this model it is assumed that planetesimals already formed in the disk and that it is accreting millimeter to decimeter sized test particles referred to as pebbles. I numerically integrate the equation of motion of a test particle including the drag force of the gas and the gravitational interactions of the planetesimal and star on this particle. The numerical implementation is thoroughly tested and compared with literature. Finally, a growth time scale is deduced for the planetesimal and I analyse how fast this body can grow through the pebble accretion mechanism.

THEORETICAL MODELS
For the protoplanetary disk model surrounding the young star, the minimum mass solar nebula model (MMSN) is used [12]. In this model it is assumed that the mass of our eight solar system planets has been spread throughout their orbital planes. The proposed model gives a valid first order approximation for the numerical implementation. The MMSN is assumed to have a dust to gas ratio of $\frac{\rho_{\text{gas}}}{\rho_{\text{dust}}} = 100$. Since the gas is partially pressure supported, a calculation reveals that it rotates at slightly less than the Kepler velocity:

$$v_g = v_k - v_{\text{hw}} \tag{1}$$

With v_{hw} defined to be the headwind velocity in a frame rotating with the Kepler velocity at 1 AU orbital distance from the star. The numerical value of the headwind can be found in table 1 [6] . Due to this deviation, particles experience a drag in the gas which can be expressed as a stopping time, which is the time needed to slow down a particle with a factor e. The stopping time can be categorized in three regimes [11], [7]:

$$t_{\text{st}} = \begin{cases} \frac{\rho_s s}{\rho_g c_g} & \text{Epstein} \\ \frac{4\rho_s s^2}{9\rho_g c_g l_{mfp}} & \text{Stokes} \\ \frac{6\rho_s s}{\rho_g |\mathbf{v}-\mathbf{v}_{\text{gas}}|} & \text{Quadratic} \end{cases} \tag{2}$$

With ρ_g the gas density, ρ_s the density of a spherical particle, $|\mathbf{v}-\mathbf{v}_{\text{gas}}|$ the difference between the particle velocity and gas velocity, c_g the sound speed, s the particle radius and l_{mfp} the mean free path of the molecules. Since the planetesimals considered are relatively inert, the orbit is assumed unperturbed circular. A local Cartesian frame is used that is rotating with the Kepler velocity. The planetesimal in the origin thus feels a headwind of v_{hw} coming towards it. For the flow pattern of the gas around the planetesimal, the analytical solution for an ideal inviscid flow is used, neglecting turbulence as it would complex matters too much. The velocity components of the flow are given by [4]:

$$\mathbf{v}_{\text{gas}} = \left[(1 - \frac{R^3}{r^3})v_{\text{hw}} \cos\theta\right] \hat{\mathbf{e}}_r - \left[(1 + \frac{R^3}{2r^3})v_{\text{hw}} \sin\theta\right] \hat{\mathbf{e}}_\theta \tag{3}$$

With R the radius of the planetesimal, v_{hw} the headwind velocity, θ the angle with the direction of the incoming flow and r the distance from the origin. The simulated flow pattern I obtained from the solution is given in figure 1. Studying the

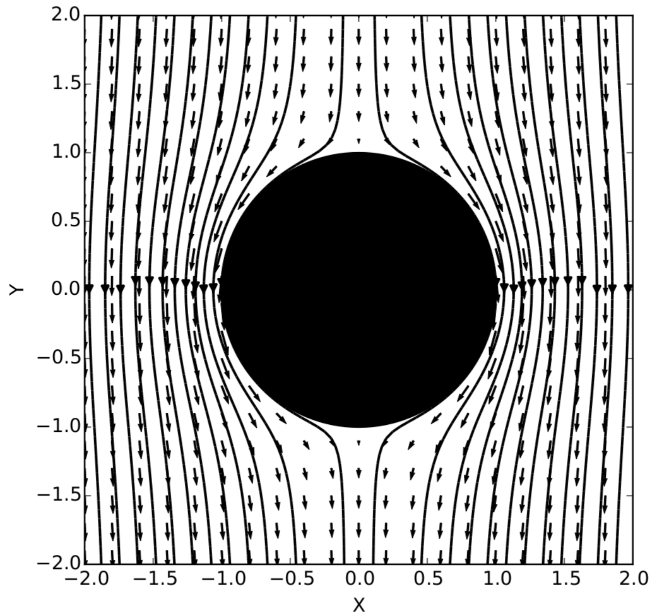

Figure 1. The ideal inviscid flow solution for the gas pattern passing a unit sphere. Turbulent movement of the flow when it passes the sphere is neglected.

gas drag in combination with gravitational influence from the star and planetesimal, requires solving the equation of motion for the pebble in the gas:

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{drag}} + \mathbf{F}_{\text{pltsml}} + \mathbf{F}_{\text{centrifugal}} + \mathbf{F}_{\bigstar} \tag{4}$$

With units of acceleration and the Coriolis acceleration and centrifugal acceleration arising from the co-rotating coordinate frame. In Cartesian coordinates the planetesimal gravity can be expressed as:

$$\mathbf{F}_{\text{pltsml}} = -\frac{GM_{\text{pltsm}}}{(x^2+y^2)^{\frac{3}{2}}}\mathbf{x} \tag{5}$$

With G the gravitational constant, M the mass of the planetesimal and $r^2 = x^2 + y^2$ the distance from the origin where the planetesimal is stationed, to a test particle. The Coriolis term can be written as

$$\mathbf{F}_{\text{Coriolis}} = -2\mathbf{\Omega}_0 \times \mathbf{v} = 2\Omega_0 \begin{pmatrix} v_y \\ -v_x \end{pmatrix} \tag{6}$$

Where $\mathbf{v}$ is the velocity of the pebble and $\mathbf{\Omega}_0$ is the Kepler frequency. I assumed $\mathbf{\Omega}_0$ points in the z-direction and that the motion is restricted to the xy-plane. The drag force can be expressed as [7]:

$$\mathbf{F}_{\text{drag}} = -\frac{(\mathbf{v} - \mathbf{v}_{\text{gas}})}{t_{\text{st}}} \tag{7}$$

Note that the velocity of the gas is also influenced by the rotation of the coordinate system. Every object rotating around a fixed axis with angular frequency Ω satisfies:

$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{\Omega} \times \mathbf{r} \tag{8}$$

Table 1. Numerical values of parameters used in calculations. The quantities dependent on disk radius are numerically evaluated on 1 AU

Parameter	Definition	Value
v_{hw}	Headwind	$5700\ \text{cm s}^{-1}$
R_{pltsml}	Planetesimal size	1 - 1000 km
ρ_{gas}	Gas density	$10^{-9}\ \text{g cm}^3$
t_{st}	Stopping time	$10^2 - 10^6$ sec
R_{pebble}	Pebble size	$0.01 - 30$ cm
$\rho_{particle}$	Particles per volume	$\frac{\rho_{\text{gas}}}{100}$
ρ_{pltsml}	Planetesimal density	$1\ \text{g cm}^{-3}$
r_0	Orbital distance	1 AU
Σ	Surface density	$1700\ \text{g cm}^2$
v_k	Kepler speed	$30\ \text{km s}^{-1}$
c_s	Sound speed	$10^5\ \text{cm s}^{-1}$
l_{mfp}	Mean free path	1.84 cm

For the gas velocity in the local frame I subtract the velocity of the rotating frame:

$$\mathbf{v}_{gas\phi} = (1-\eta)\,\Omega r \mathbf{e}_\phi - \Omega_0 r \mathbf{e}_\phi \tag{9}$$

It can be shown that in the local frame, this leads to the simplified expression:

$$\mathbf{v}_{\text{gas}} \approx \left(-\frac{3}{2}\Omega_0 x - v_{\text{hw}}\right)\mathbf{e}_y \tag{10}$$

Where I found the Kepler shear corrected gas velocity. The last two terms of equation 4 can be combined to the tidal acceleration:

$$\mathbf{F}_{\bigstar} + \mathbf{F}_{\text{Centrifugal}} = 3x\Omega_0^2 \mathbf{e}_x \equiv \mathbf{F}_{\text{Tidal}} \tag{11}$$

Where I used that $x, y \ll r_0$ with r_0 the distance of the solar center to the planetesimal. All the forces are expressed in Cartesian coordinates and need to be integrated to obtain the particle trajectories.

NUMERICAL IMPLEMENTATION

The equation of motion is integrated using Runge Kutta Fehlberg variable step techniques[1]. The pebbles start at an initial y-coordinate y_s far enough from the planetesimal. The used numerical values of parameters are defined and given in table 1 To determine the amount of particles being swept up by the planetesimal per unit time, it is useful to start with the impact parameter b:

$$b = \frac{x_2 - x_1}{2} \tag{12}$$

Where x_2 and x_1 are the positions of the last hitting pebble and the first hitting pebble respectively, see figure 2. The total flux through this region is just:

$$F_{\text{coll}} = A\mathbf{v} \cdot \hat{\mathbf{n}} \tag{13}$$

With $A = \pi b^2$ the surface of the region, $\hat{\mathbf{n}}$ a vector perpendicular to this surface and $\mathbf{v}$ the velocity of the pebble. This gives the expression for the flux:

$$F_{\text{coll}} = \pi b^2 v_{\text{hw}} \tag{14}$$

To quantify this area of influence, I look at the collisional fraction. I define this to be the ratio of the impact flux and

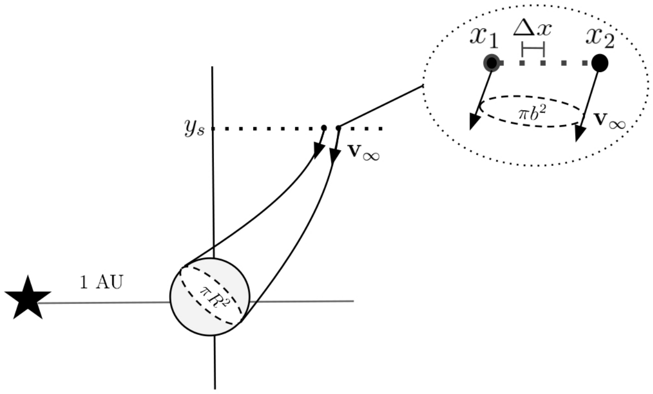

Figure 2. A schematic overview of the quantification of the accretion process. The pebbles initially start far away from the planetesimal (in the origin) to be unperturbed at the initial coordinates. From the first and last hit a collisional cross-section can be determined.

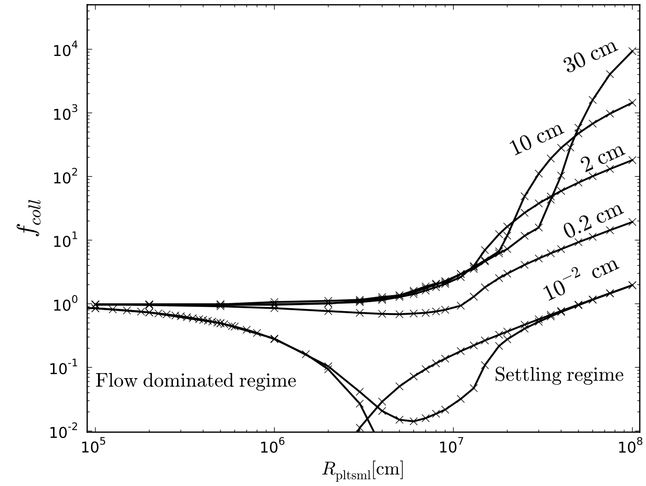

Figure 3. The collisional fraction $f_{\rm coll}$ as function of planetesimal size. There is a barrier around 100 km planetesimal size, sweeping up pebbles < 3 mm. For larger pebble sizes no minimum occurs. For smaller particles (< 0.2 cm), the accretion starts in the flow dominated regime, enters the barrier and transitions to settling directly. For particles > 0.2 cm, it starts in the geometric regime, there is no barrier anymore and the settling regime is reached the same way. For higher particle size $>$ 2 cm, gravitational focusing kicks in first and fast settling is characterized by the sharp transition.

the flux imagined going through the geometric cross-section of the planetesimal:

$$f_{\rm coll} = \left(\frac{b}{R_{\rm pltsml}}\right)^2 \tag{15}$$

To compare the growth rate of the planetesimal with the lifetime of the disk, I need to derive a growth timescale. In the derivation I assume that collisions with the planetesimal are sticky, this is a valid assumption for the considered parameter space [9]. In this case, the rate of change of the mass is just what comes through the impact surface (eq: 14) multiplied by the density of pebbles:

$$\dot{M} = F_{\rm coll}\rho_{\rm part} \tag{16}$$

This can be converted to a characteristic growth time:

$$t_{\rm growth} = \frac{M}{\dot{M}} \tag{17}$$

Where $t_{\rm growth}$ is defined as the time needed for the planetesimal to e-fold its mass.

NUMERICAL RESULTS

The collisional fraction as a function of planetesimal radius is shown in figure 3. These results are obtained with the potential flow solution in accordance with non-viscous flow[4]. The collisional fraction can be categorized in three regimes:

1. The geometric regime, $f_{\rm coll} = 1$
2. The flow-dominated regime [2], $f_{\rm coll} < 1$
3. The settling regime [7], $f_{\rm coll} > 1$ [1]

The first regime shows accretion in which the impact parameter b is approximately equal to the planetesimal cross section. In the second regime the pebbles hardly fall on the sphere because they are coupled too much to the gas, this is called the flow-dominated regime. In the latter regime gravitational focusing in combination with the gas drag increases the impact parameter. The gas drag is responsible for lowering the momentum of the incoming particle. This in turn makes a collision more probable with the planetesimal since energy is not conserved anymore due to friction. On the other hand, if only gravitation was present, conservation of energy would prevent point particles from colliding in close encounters. For gravity dominated but gas drag mediated interactions, the physical body of the planetesimal can be shrunk down to a point particle and gravitational focusing will still be approximately equal. This is where the settling regime starts.

In all cases for 1 km planetesimals the collisional fraction starts at the geometric cross-section (geometric regime). The collisional fraction goes down significantly in the flow-dominated regime since particles are highly coupled to the gas. Eventually the 10^{-2} and 0.2 cm curves have a minimum at around $R_{\rm pltsml} \approx 100$ km. This shows that a finite barrier occurs when the particles being accreted are too small. When the planetesimals become bigger, gravity dominates (settling regime) and the barrier disappears. For pebbles larger than 0.2 cm the cross-section exceeds the geometrical cross-section and there is no minimum anymore.

The collisional fraction is converted to a growth timescale using equation 17, see figure 4. An expression for the growth timescale in the settling regime was obtained already by Ormel and Klahr (2010) [7]:

$$t_{\rm growth} \approx \frac{1}{4\pi G\rho_{\rm part}t_{\rm st}} \tag{18}$$

[1] There is a small catch here. The settling regime is referred to as the regime where particles reach terminal velocity. This characterizes rapid accretion. However, the smallest particles (0.01 cm) enter the settling regime directly after the deflection, therefore concluding that the settling is not always the regime where $f > 0$. For details on the nomenclature of slow settling and fast settling, see [10]

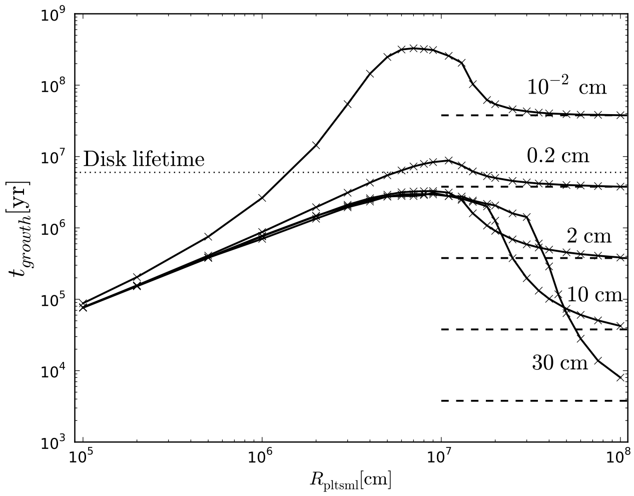

Figure 4. The growth rate for planetesimals in the given range as a function of planetesimal size and a pebble density of $\frac{\rho_{\rm gas}}{100}$, following from the dust to gas ratio of the MMSN. The dashed lines represent the constant lines when filling in the corresponding stopping time in equation 18.

Where $\rho_{\rm part}$ is the number of particles per unit volume. The value follows from the gas to dust ratio in the MMSN. The expression given in equation 18 is independent of the planetesimal properties and gives the constant the growth time must approach in the settling regime. The dashed lines correspond with this settling constant corresponding to the particle size/stopping time. The smaller the test pebble size, the larger the growth barrier. The barrier is identified at around 100 $\rm km$ planetesimal radius. It can be seen that for the largest three pebble sizes, curves are crossing each other. Since the stopping times are increasing, settling becomes harder for these pebbles because they are not slowed down enough within the time they go past the planetesimal. However, when the gravity of the planetesimal becomes large enough, the process speeds up again and the pebbles settle eventually, explaining these intersections.

CONCLUSIONS

I have investigated with a numerical simulation whether planetesimals in the range of 1-1000 $\rm km$ in an unperturbed Keplerian circular orbit at a distance of 1 AU from the star are able to grow by sweeping up pebbles in the range of 0.01 $\rm cm$ to 30 $\rm cm$. I conclude from the growth rate of the planetesimals that there is a barrier between the flow-dominated regime and the settling regime until the test pebble size reaches around 1 $\rm cm$. In this region, pebble accretion kicks in and growth becomes easier. Using an optimistic typical disk lifetime of $t_{\rm disk} \approx 6$ Myr, full growth can be realized for $R_{\rm pebble} \geqslant 2$ cm. The barrier is identified around 100 $\rm km$ planetesimal size for each pebble size. While this barrier can be a bottleneck for growth, the barrier is not infinitely large. The bottleneck radius corresponds roughly with the radius of asteroid belt objects. A smooth transition has been found between the two regimes and gravity will take over at a certain point. One possible scenario could be that growth stops when the barrier has been reached and that another mechanism takes over from here that ensures $t_{\rm growth} < t_{\rm disk}$.

ROLE OF THE STUDENT

The research was proposed by supervisor Dr C.W. Ormel. The research contained a literature study on the subject, a mathematical analysis of existing concepts and a numerical implementation of the accretion model and the interpretation of new results. This whole process was fully carried out by the student. The paper for the SRC application was fully written by the student.

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