Pessimistic Equilibrium Flow and Its Network Design Implications

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Quality of service in urban transportation network is very important to the users of this network. Travel time in street networks is one, very important, aspect of this service, particularly for trips with purposes such as work or school. A conservative route choice concept based on a pessimistic view of travel time has been proposed to render a pessimistic equilibrium flow (PEF). This flow is different from a user equilibrium flow (UEF), particularly in congested networks, and has sometimes a lower total travel time than UEF. A street network design problem has been proposed to take into account the travel time fluctuations (or service quality) in the network. This design model is based on PEF and is called pessimistic network design (PND) model. It is shown by an example network that an implication of PEF consideration in a PND model is that projects with lower potential in getting congested are chosen as compared with the conventional network design problem.

1. Introduction

Statistical analyses of traffic characteristics show that these characteristics may vary significantly at any given point in a street network and specific time of day (Smeed and Jeffcoat, 1971). This variation, or fluctuation, in flow characteristics is a result of fluctuations in O/D travel demand and various other stochastic events affecting street capacities (Iida and

Wakabayashi, 1989). One such, and most important, flow characteristic is passenger travel time. Research results suggest normal and lognormal distributions for the travel time of the users of the network (Taylor, 1982). Moreover, past theoretical and statistical analyses of the problem attempt to estimate the variance of the user travel time as a function of the respective average and free flow travel time (see also Taylor, 1982). This variance tends to increase as the value of travel time increases.

Work and school are two major and important trip purposes in morning peak. To be "on time" is a common characteristic of these daily trips, which is a cause of worry for the travelers with these trip purposes, particularly in large metropolitan areas. Frequent delays may result unbearable cost. Thus, in such trips travelers act conservatively, and try to reach destinations on time, or before the time.

Passenger conservatism in this respect is not far from pessimism with regard to the network performance. Thus, in order to consider the passenger behavior to overcome the daily delay worries, one may postulate the following hypothesis in route choice: "Passengers choose routes with minimum worst case travel time," (and not the average travel time). We refer to this concept as Pessimistic Route Choice (PRC). This concept considers both the average measure, and a measure of the variation of the travel time in the travelers' decisions. This is in contrast with the conventional route choice behavior which is based on the average travel time only.

To elaborate more on this concept, let us consider the following example. Figure 1 shows two routes that may be used by a traveler from home to work using automobile: a direct route (d) which takes 10 to 50 minutes; and an indirect route (i) which takes 30 to 35 minutes. Which route will be chosen by this traveler? A pessimistic traveler (by virtue of his/her trip purpose) would choose i, because in this case it is enough to start the trip 35 minutes before the work starting time (instead of 50 minutes for route d) to avoid any delay (assume that starting the trip before the worst case travel time of a chosen route would serve no purpose).
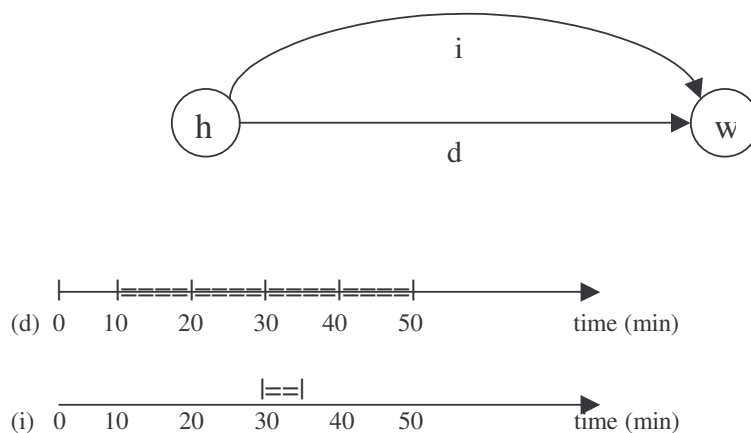

Figure 1. Pessimistic route choice concept

Considering PRC, and assuming inelastic morning peak demand (with respect to travel time), one may compute the lost travelers' time (LTT) as:

$$LTT = \sum_{(k,s)\in P} d^{ks}\hat{t}^{ks} \tag{1}$$

where d^{ks} is the demand (rate) from origin k to destination $s, (k, s)\in P$, where P is the set of origin-destination (O/D) pairs. $\hat{t}^{ks}$ is the expected shortest passenger travel time from k to s with traffic fluctuation effects.

The purpose of this study is to (a) define the Pessimistic Equilibrium Flow (PEF) problem in urban networks, and (b) discuss some network design implications of the flow problem under travel time fluctuation, and to show that this phenomenon is an important factor with a potential to alter the results (the selected projects) of the conventional network design (ND) problem. The scope of the paper is limited to static transportation network flow with inelastic demand, but randomly distributed path/link travel time.

Section 2 of the paper reviews some of the previous works in the scope of the purpose of this paper. Section 3 introduces the pessimistic equilibrium flow problem, followed by Section 4 which presents a network design problem under travel time fluctuations analogous to the conventional network design problem. Section 5 is devoted to a discussion regarding PEF and several numerical analysis and justifications. Section 6 discusses the results, and suggests avenues for future research.

2. Previous endeavors

The traffic fluctuation phenomenon in transportation network flows has been analyzed under topics such as "stochastic traffic assignment," and "reliability" in street networks.

Herman and Lam (1974) have studied the variability of travel time for certain trips made by automobiles. They assumed that travel times on different sections of a path in the network are independent from each other, and that travel times on sections with equal lengths are identically distributed. Their analysis of the problem offered the following relationship, based upon the above assumptions:

$$S = \gamma\sqrt{t} \tag{2}$$

where t and S are the average value and standard deviation of the travel time, respectively, and γ is a constant. Based on certain statistical analysis (see Smeed and Jeffcoate, 1971), they concluded that the variability of travel time of a route may be well represented by a normal distribution. Taylor (1982) used a type of equation (2) in a study of public transportation travel time variability. Taylor (1999) used equation (2) in a stochastic traffic assignment model (called "Traffik Plan"), and tried to estimate γ by gathering field data from a street network.

Richardson and Taylor (1978), using statistical data, presented and studied the following expression for the route travel time:

$$S = \psi(t/_{t^\circ} - \beta)\sqrt{t} \tag{3}$$

where s and t are as defined before, and $t°$ is the free flow (route) travel time; and ψ and β are two parameters of the expression.

In the area of reliability in street networks, Iida and Wakabayashi (1989) have noted the importance of transportation networks in today's large and complex societies, and the necessity of having reliable street networks which offer alternative routes to the users in cases of the failure of parts of the network because of accidents, maintenance, congestion, and natural incidents (earthquakes, floods, etc.). This reliability index is called "terminal reliability," and this measure for an O/D (k, s) in a street network is the probability of having at least one path from k to s such that it can offer service at a specified level during a period of time. Their problem is to compute the probability of having at least one non-congested path from k to s. Such non-congested path is a path in which all links are non-congested. They, then, offered a heuristic algorithm, called "intersection method" which takes advantage of limited number of minimum path and cut sets, and presents a good estimate of the terminal reliability in a street network.

Bell et al. (1999) emphasize the need to develop tools for evaluating the impact of stochastic incidents upon the street networks, having noted the importance of the management of demand and supply in urban transportation. They suggest the following two important network performance measures for this purpose: (a) travel time of different O/D paths in the network, and (b) the expected shortest travel time between various O/D pairs. The importance of these measures stems from the fact that a good performance of a street network translates into being able to transport passengers from their origins to the respective destinations within an acceptable duration of time. These researchers concluded that transportation demand and link capacities are stochastic variables, and (for simplicity) assumed that trip demand is normally distributed in a specific period of time. They, then, using two examples explain how to get path travel time distribution and the expected shortest travel times between O/D pairs. The results obtained from these two examples show that variation of path travel times of congested links is high, and such paths are not reliable.

Lee et al. (2000) have analyzed congestion and passenger travel time reliability in a network from the stand point of stochastic variation in the capacity of streets in the network. They noted that when capacity of a street falls below the flow in that street, its congestion and travel time variability increases. They postulated that passengers choose routes not with a minimum value of travel time, but with a minimum travel time variability. Based on this hypothesis, they presented a model for traffic assignment, and show the ability of this model in solving large scale problems, by solving some examples. It must be emphasized that the model presented by Lee et al. (2000) suffers a structural drawback in modeling the behavior of passengers in choosing paths, by ignoring travel time, and considering only its variability.

Chen and Recker (2000) emphasize that understanding the underlying mechanisms and determinants of route choice would affect our ability to estimate network flows better, and would influence our decision to design the network to perform better. They rate our understanding of the links between network performance, demand and supply variations, and the resultant traveler behavior as poor. They tried to capture an understanding of the risk taking behavior in route choice, and its impact on the travel time reliability in cases where demand and supply vary. They, also, report examining network performance under different route choice models.

Lo and Tung (2003) tried to model network performance when link capacities are subject to travel time variability. They postulated that drivers would select routes to lower travel time

variability. They defined a probabilistic user equilibrium condition for each O/D pair, which were formulated as two (reliability) constraints. These constraints are added to a network capacity model. However, the constraint set of this model is not convex, thus there is no guarantee for a solution found to be globally optimum. Moreover, having formulated the problem in path flows, there is the difficulty of working with excessive path variables in real networks.

Yin et al. (2004) recognizing that commuters' response to travel time fluctuation is a base for the evaluation of intelligent transportation systems, tried to formulate departure time and route choice process under uncertainty. They presented an expected travel disutility for the commuters' departure time and route choices as a function of travel delay, travel time uncertainty, and early or late arrival penalty. They presented a simultaneous route and departure-time user equilibrium, formulated as a non-linear complementarily problem, and converted into an unconstrained minimization problem to be solved by Nelder-Mead simplex method (See the article for a reference of this method).

Clark and Watling (2005) propose a technique for estimating the probability distribution of the total travel time in the cases where frequent events of various types affect the operation of the network. Examples of these events have been introduced as accidents, parking violations, and traffic signal failure (on the supply side); snow, and flooding (on the environment side); and daily variation in activity pattern (on the demand side) of the problem. They have computed moments of the total travel time distribution analytically. Thus, a density function is fitted to these moments, and a measure of unreliability of the network has been defined based on this function.

The above discussion convey sufficient evidence that current literature of transportation is aware of the importance of travel time variability in route choice. (see also Bell and Cassir, 2000). This effect would, then, necessitate re-evaluation of current traffic assignment routines, as well as the formulation of the network problems which rely upon them, such as network design problem.

3. Notations, assumptions and definitions

Let $N(V, A)$ be a network with V as the set of nodes and A as the set of links. Let n be the number of nodes, $n = |V|$, k and s represent the origin and destination, respectively. Let, also, P denote the set of O/D pairs (k, s), with demand d^{ks} from k to s.

Moreover, let ρ denote a path in the network, and ρ^{ks} the set of paths from k to s. x_ρ^{ks} is the flow in path ρ from k to s, and x_{ij} the flow in link (i, j) which experiences the average travel time t_{ij}. The travel time of link (i, j) is a convex function of the flow in that link only. Let us denote by t_ρ^{ks} the average travel time in path ρ from k to s, $\rho \in \rho^{ks}$, and $(k, s) \in P$.

Furthermore, suppose that A_y is a set of projects under considerations. Project link (i, j), $(i, j) \in A_y$, has a cost of e_{ij}. B is the budget, and represents the limit to the expenditure in the network. $y_{ij}, (i, j) \in A_y$, is the network design variable, taking a value of 1 if project (i, j) is selected to be implemented, otherwise 0.

Assume that: (a) projects are to be evaluated based on morning peak demand. (b) In peak demand period all trips are either work trips, school trips, or trips that are crucial to be

finished on time. (c) Link travel times in the network are stochastic variables with normal distributions, which are independent from each other. For link (i, j), the standard deviation of travel time is given as follows:

$$S_{ij} = \psi\left(t_{ij}/t_{ij}^\circ - \beta\right)\sqrt{t_{ij}} \tag{4}$$

where all variables and parameters are as defined in equation (3), but here for link (i, j). It follows that the path travel times in the network are stochastic variables, each of which being distributed normally, with mean $t_\rho^{ks} = \sum_{(i,j)\in\rho} t_{ij}$ and variance $S_\rho^2 = \sum_{(i,j)\in\rho} S_{ij}^2$, $\rho \in \rho^{ks}$, and $(k, s) \in P$.

Based on assumptions (b) and (c), (d) assume that travelers in the network demand to reach their destinations on specific times, at least with a high probability. For example, they require to reach their destinations before those specific times for $\alpha\%$ of the time. Thus, they look at the path travel times pessimistically (or conservatively), and their expected (conservative) path travel times are those that exceed the respective actual ones for most (e.g., $\alpha = 95\%$) of the times. To clarify this point, let us consider a path from an origin to a destination, with a travel time density function as shown in figure 2. To answer the question that when should a traveler in this path start the trip to reach the destination on time with a probability of, say, 0.95, one may compute the expected (conservative) path travel time $(\hat{t})$, as $\hat{t} = t + K_\alpha S$. Thus, the traveler should start the trip $\hat{t}$ units of time ahead of the desired time. (Then, if the traveler reaches destination at time $t^a, t^a < \hat{t}$, we assume $\hat{t} - t^a$ will be lost.). To specify an average, or a representative, value of α for a case, requires an independent study or opinion survey. To be more specific, and for numerical examples, a value of $\alpha = 95\%$ will be used in what follows.

We, now, state our next assumption (or hypothesis) that (e) passengers choose paths according to their least expected (conservative) travel times. That is, the expected conservative travel time of the selected path by a traveler from an origin to a destination is minimum among all available alternative paths. The resulting flow will be called pessimistic equilibrium flow or *PEF* in short.

To summarize: given the current O/D demand for the peak period, which is a result of a multitude of decisions on the part of many trip-makers (regarding making the trip, time of starting the trip, choice of destination and mode of travel), the choice of route is based on a pessimistic attitude which guarantees on time arrival at the destination for $\alpha\%$ of the time. Other criteria of route choice are assumed to be negligible due to the prevalence of the pessimistic attitude in reaching destinations for the serious morning peak trip purposes such as work or school. Of- course, the planned time affects the departure time decisions resulting the current demand (d^{ks}).
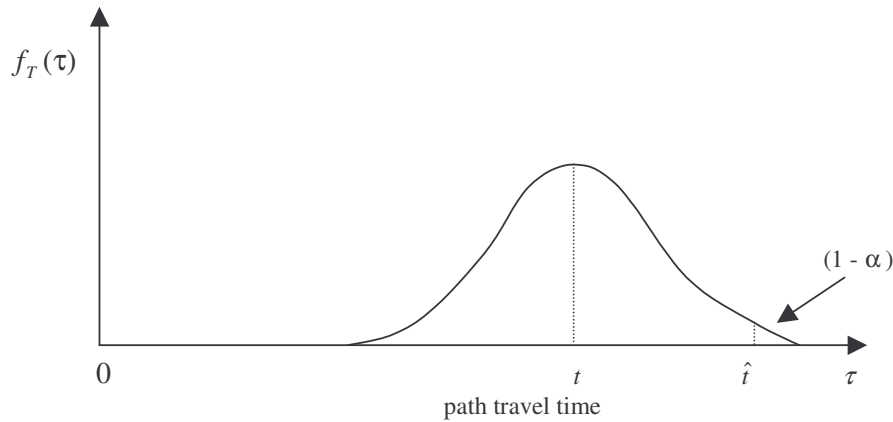
Figure 2. Path travel time density function

PEF problem, as defined above, is a flow problem in a network where the link cost function is a function of flows in that and other links through the variance of the path travel times, S_ρ^2. This problem may be formulated as a non-linear complementarily problem, which may be stated as follows (Aashtiani and Magnanti, 1981):

$$(PEF) \qquad \hat{t}_\rho(x) - \hat{t}^{ks} \geq 0, \qquad \forall \rho \in \rho^{ks}, \forall(k,s) \in P$$

$$x_\rho^{ks}[\hat{t}_\rho(x) - \hat{t}^{ks}] = 0, \qquad \forall \rho \in \rho^{ks}, \forall(k,s) \in P$$

$$\sum_{\rho \in \rho^{ks}} x_\rho^{ks} = d^{ks}, \qquad \forall(k,s) \in P$$

$$x_\rho^{ks} \geq 0, \hat{t}^{ks} \geq 0, \qquad \forall \rho \in \rho^{ks}, \forall(k,s) \in P$$

Note that $\hat{t}_\rho(x)$ is a strictly monotonic function of x (see Lemma 1 in the Appendix A), which is a requirement for the existence and uniqueness (in link flows) of the solution to the above problem.

In this view, any algorithm which solves the respective general *UEF* problem or its extensions to model elastic demand case or joint traffic assignment and other travel decisions, as well as algorithms which are devised to solve various other problems which are based upon *UEF* such as bi-level programming problems (including network design problem) may be exploited (with some modifications) to incorporate *PEF* instead of *UEF* for morning peaks.

4. A network design problem under travel time fluctuation

To define a network design problem under travel time fluctuation, let us start with the definition of the Conventional (fixed demand) Network Design problem (*CND*) as follows:

(CND) $Min \ Z = \displaystyle\sum_{(i,j)\in A \cup A_{y1}} x_{ij}^* t_{ij}(x_{ij}^*)$

       $s.t.: (1)$ $\displaystyle\sum_{(i,j)\in A_y} e_{ij} y_{ij} \leqslant B$

           (2) $y_{ij} = 0/1, \quad \forall(i,j)\in A_y$

           (3) x^* *is the user equilibrium flow in* $N(V, A \cup A_{y1})$.

where $A_{y1} = \{(i,j) : (i,j) \in A_y, y_{ij} = 1\}$.

In this problem, it is intended to choose a subset of budget feasible projects from among a set of available projects which minimizes the total users' cost in the network, Z. x_{ij}^* is the user equilibrium flow in link (i, j) of the selected network, when there is no travel time fluctuation.

The (CND) objective function, Z, may be a good indicator of the costs that are imposed upon the society, including users' cost (travel time, vehicle depreciation, etc.), costs of limited resources (fuel, etc.), and environmental costs (pollution, accidents, etc.). However, this model is incapable of taking the effect of travel time fluctuations upon route choice into account. According to what has been said in section 1 of this paper, Z in (CND) problem is not a good representation of the travelers' time cost. This cost (that a fraction of which might be lost at the destinations) is rather $\displaystyle\sum_{(k,s)\in P} d^{ks}\hat{t}^{ks}(x^{**})$, where x^{**} is the *PEF* in the network.

Hence, one may propose the following network design problem for the case of travel time fluctuations:

(PND) $Min \ Y = \displaystyle\sum_{(i,j)\in A \cup A_{y1}} x_{ij}^{**} t_{ij}(x_{ij}^{**}) + \theta \displaystyle\sum_{(k,s)\in P} d^{ks}\hat{t}^{ks}(x^{**})$

       $s.t.: (1)$ $\displaystyle\sum_{(i,j)\in A_y} e_{ij} y_{ij} \leqslant B$

           (2) $y_{ij} = 0/1 \quad , \forall(i,j)\in A_y$

           (3) x^{**} *is pef in* $N(V, A \cup A_{y1})$.

The first part of the objective function in the above problem (call it xt) represents the social cost paid in the network (excluding travelers' time cost) for transporting passengers from origins to destinations, and the second part (call it dt) is the travelers' time cost incurred to the users of the network to avoid anxiety or worry for the most part. θ is a weighting factor, which also homogenizes the units of the two parts of the objective function. A convex combination of xt and dt may be written as $\eta \, xt + (1-\eta) \, dt$, for $0 \leqslant \eta \leqslant 1$, Then, if one defines θ as $(1-\eta)/\eta$ for $\eta \neq 0$, one may ensure that xt appears once in its integrality in the design objective function, although one may feel free to choose θ to increase the weight of the consumer surplus-related, or worry-related, part of the function.

A crucial part of any algorithm to solve (PND) is a solution procedure to find x^{**} for any network $N(V, A \cup A_{y1})$, which is constructed by following a decision vector y with elements y_{ij} . Size of y is $|A_y|$. The following is one such procedure which solves the *PEF* problem heuristically. It is of incremental nature.

5. Presentation and discussion of a *PEF* algorithm

Superscript m represents iteration number, and $\breve{x}^{ks}$ denotes the assigned part of (k,s) O/D demand to the network. It is assumed that τ_ρ^{ks}, the random variable of travel time in path ρ from k to s is distributed $N(t_\rho^{ks}, (S_\rho^{ks})^2)$. Define Δx^{ks} as $\frac{1}{D}$ of d^{ks}, where D is an appropriate integer number. And, finally H^m is the set of remaining O/D with $\breve{x}^{ks} < d^{ks}$ in iteration m.

Step 0.   (initialization). $m := 1; t_{ij}^m := t_{ij}^\circ, \forall(i,j) \in A; \breve{x}^{ks} := 0, \forall(k,s) \in P, H^m = P.$

Step 1.   Choose (k,s) randomly. Check if $\breve{x}^{ks} < d^{ks}$, otherwise repeat this step.

Step 2.   For all links $(i,j) \in A$, compute an estimate of the standard deviation of the travel time of link (i,j) in iteration m as follows (ψ and β are two constant known parameters):

$$S_{ij}^m = \psi \, (t_{ij}^m / t_{ij}^\circ - \beta)\sqrt{t_{ij}^m} \tag{5}$$

Step 3.   Find $\rho^*, \rho^* \in \rho^{ks}$, such that $\hat{t}_{\rho^*}^{ks,m}$ is minimum. ($\hat{t}_{\rho^*}^{ks,m}$ is the minimum expected (conservative) path travel time.)

Step 4.   Set $m := m+1$, and assign another increment of O/D demand, Δx^{ks}, to the network as follows:

$$x_{ij}^m = x_{ij}^{m-1} + \Delta x^{ks}, \forall(i,j) \in \rho^*, \rho^* \in \rho^{ks} \tag{6}$$

$$\breve{x}^{ks,m} = \breve{x}^{ks,m\ 1} + \Delta x^{ks}$$

If $\breve{x}^{ks,m} \geqslant d^{ks}$, then $H^m = H^{m-1} - \{(k,s)\}$.

Step 5.   Update the travel times:

$$t_{ij}^m := t_{ij}(x_{ij}^m), \quad \forall(i,j) \in \rho^*, \rho^* \in \rho^{ks}$$

Step 6.   Stop, if $H^m = \Phi$, the empty set; otherwise go to step 1.□

Remark 1. $\hat{t}_{\rho^*}^{ks,m}$ in step 3 of the above algorithm may be computed by using a Dijkstra- type shortest path routine, modified as follows. Let $\hat{t}(j), t(j)$, and $S(j)$, be the current conservative shortest time, the current average shortest time, and the current standard deviation of travel time, from origin k to node j, respectively. Let, also, $l(j) = i$ denote the node preceding node j in the current path with minimum expected (conservative) travel time.

Step 3.0 $t_{kj} := t_{kj}$ if link (k,j) exists, otherwise $t_{kj} = +\infty$ (a large number, e.g. sum of n top t_{ij} values). $N := V - \{k\}, t(k) := 0, S(k) := 0; \hat{t}(j) := t_{kj} + K_\alpha S_{kj}, l(j) = k,$ for all $j \in N. c := 1$ (c is a counter).

Step 3.1   Find i^* such that $\hat{t}(i^*) = Min_{j \in N}\{\hat{t}(j)\}$. Set $N := N - \{i^*\}, c := c+1$.

Step 3.2   For all links $(i^*, j), j \in N,$ if

$$\hat{t}(j) > (t(i^*) + t_{ij}) + K_\alpha(\sqrt{S^2(i^*) + S_{i^*j}^2})$$

then, set: $t(j) := t(i^*) + t_{ij}$, $S(j) := \sqrt{S^2(i^*) + S^2_{i^*j}}$, $\hat{t}(j) := t(j) + K_\alpha S(j)$.

$l(j) := i^*$. If $c < n$, go to Step 3.1; otherwise continue.

Step 3.3.   Construct the shortest expected (conservative) tree, as:
$T = \{(l(j), j) | j \in V, j \neq k\} \square$.

Remark 2. In the above discussion it is assumed that for any $(k, s) \in P$ travelers choose routes based on $\hat{t}_\rho = t_\rho + K_\alpha S_\rho$, where $S^2_\rho = \sum_{(i,j) \Subset \rho} S^2_{ij}, \rho \in \rho^{ks}$. Thus, S_ρ's are interdependent variables necessitating the heuristic procedure given above to incrementally updating the estimates of S_ρ in step 3 of the *PEF* algorithm. Aashtiani and Magnanti (1982) present a more exact treatment of problems similar to *PEF* problem, where link travel cost is function of flows in other links as well as its own. Such problems may be written in terms of complementary slackness problem, and solved by the related algorithms.

Now, suppose that choice of route is based on the route cost $\tilde{t}_\rho = \sum_{(i,j) \Subset \rho} \tilde{t}_{ij}(x_{ij})$, where

$\tilde{t}_{ij}(x_{ij}) = t_{ij}(x_{ij}) + K_\alpha S_{ij}$.

Lemma 2 in Appendix A shows that $\tilde{t}_{ij}(x_{ij})$ is strictly convex. Then *PEF* problem may be defined as follows:

$$(PEF1) \qquad Min \sum_{(i,j) \in A} \int_0^{x_{ij}} \tilde{t}_{ij}(u)du$$

$$s.t.: \sum_{\rho \in \rho^{ks}} x^{ks}_\rho = d^{ks}, \qquad\qquad \forall (k, s) \in P \qquad\qquad (7)$$

$$x^{ks}_\rho \geqslant 0, \qquad\qquad \forall \rho \in \rho^{ks}, \forall (k, s) \in P \qquad\qquad (8)$$

$$x_{ij} = \sum_{(k,s) \in P} \sum_{\rho \in \rho^{ks}} \delta^{ks}_{ij,\rho} . x^{ks}_\rho, \qquad\qquad \forall (i, j) \in A \qquad\qquad (9)$$

where $\delta^{ks}_{ij,\rho} = 1$ if $(i, j) \in \rho, \rho \in \rho^{ks}$, and 0 otherwise. Any convex programming algorithm may be exploited to solve this problem (see e.g. Sheffi, 1985). It may, also, be shown that at the point of equilibrium all used paths between an O/D pair (k, s) have equal travel costs $\tilde{t}^{**}_\rho = \sum_{(i,j) \Subset \rho} t_{ij}(x^{**}_{ij}) + K_\alpha \sum_{(i,j) \Subset \rho} S_{ij}(x^{**}_{ij})$, which is less than that of an unused path. Then, all properties of *UEF* problem follows for *PEF*; all extensions of *UEF* apply for *PEF*; and all solution procedures of *UEF* may be used to solve the respective problems of *PEF*.

If users of the network think link-wise, like in problem *PEF*1, Lemma 3 in Appendix A shows that they levy higher mental tolls upon themselves than if they think path-wise, like in problem *PEF*.

5.1 A numerical example for *PEF*

To show the pessimistic equilibrium flow (*PEF*) in a network, and to see how it differs from a conventional user equilibrium flow (*UEF*), let us analyze the network in figure 3 with 2 nodes and 2 links. The travel time functions are given for the two links in the network. Link 1

is a short link with low capacity, while link 2 to the contrary is longer and has higher capacity. Demand from $k = 1$ to $s = 2$ is q vehicles per hour.
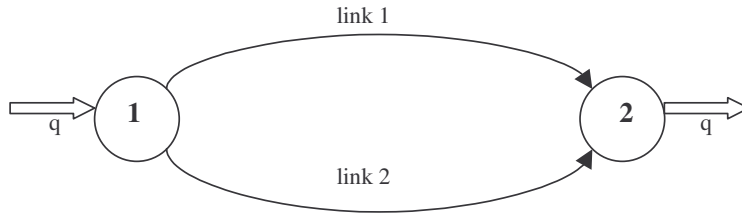
$$t_1 = 0.05 + 9.0 \times 10^{-6} \left(\frac{x_1}{100} \right)^4, \quad t_2 = 0.12 + 1.0 \times 10^{-6} \left(\frac{x_2}{100} \right)^4 \quad \text{in hours}$$

Figure 3. An example network to compare PEF and UEF.

Using *UEF* and *PEF* algorithms, the results of the two traffic assignments are given in table 1 for 4 levels of q. For the latter algorithm, the standard deviation of link travel time is assumed to be as follows:

$$S_l = 0.2 \left(\frac{t_l}{t_l^\circ} - 1 \right) \sqrt{t_l}, \qquad l = 1, 2$$

In fact, for this simple network, the solutions may be obtained by the Kuhn-Tucker conditions of optimality and the constraint set. The reader may verify that for the *PEF* problems, $\hat{t}_\rho^{**}$ for the used paths (or links, in this example) between $k = 1$ and $s = 2$ are equal, and less than this value for the unused path (link).

Table 1. The results of UEF and *PEF* traffic assignments*.

q	Conventional Traffic Assignment			Pessimistic Traffic Assignment				
	Link 1 $\left(\begin{array}{c} vol. \\ tt. \end{array} \right)$	Link 2 $\left(\begin{array}{c} vol. \\ tt. \end{array} \right)$	Veh- hr	Link 1 $\left(\begin{array}{c} vol. \\ tt. \end{array} \right)$	Link 2 $\left(\begin{array}{c} vol. \\ tt. \end{array} \right)$	Veh-hr traveled	Veh-hr planned	% increase
1	2	3	4	5	6	7	8	9
8	8.00 0.086 9.40	0.00 0.120 0.60	0.6949	8.00 0.086 8.90	0.00 0.120 1.10	0.6949	0.7538	8.5
10	0.120 9.45	0.120 5.55	1.2020	0.106 9.00	0.120 6.00	1.0796	1.2020	11.3
15	0.121 9.80	0.121 10.20	1.8220	0.109 9.40	0.121 10.60	1.7090	1.8190	6.5
20	0.130	0.130	2.6370	0.123	0.133	2.5360	2.6640	5.0

*q is in 100 of vehicles per hour, travel time (tt) is in hour, volume (vol.) is in 100 of vehicles per hour, vehicle- hour (veh-hr) traveled, and the veh-hr planned are in 100 veh-hrs.

This table shows that when q is low (say 5) the two algorithms give similar flow patterns, however as q rises, and congestion appears in the network, these two patterns become different. One interesting result in table 1 is the lower value of veh- hr traveled for PE as compared to UE. This is, of course, not surprising, because in UE flow problem vehicle- hour is not minimized while in PE flow problem the second part of the objective function is sensitive to the congestion and tries to avoid assignment of flow to paths that have congested links: Increase of travel time in a link increases the standard deviation of the travel time in that link, and thus those of the paths comprising that link. This makes the flows in these paths to reduce. This is an effect similar to that of the tolls $\left(x_{ij}.dt_{ij}(x_{ij}) \middle/ dx_{ij} \right)$ in the marginal cost function of links in the system equilibrium flow problem.

Column 7 of table 1 shows the vehicle- hour traveled in the network under pessimistic rule of traffic flow ($\sum_{(i,j)\in A} x_{ij}^{**} t_{ij}(x_{ij}^{**})$), and column 8 of this table shows the respective time assigned for this purpose ($\sum_{(k,s)\in P} d^{ks} \hat{t}^{ks}(x^{**})$). Column 9 of this table shows that, on the average about 7 to 8 percent of the total veh-hr assigned for travel is not actually used for this purpose (i.e., travel), but set aside to be on time most of e.g., $\alpha = 95\%$ of the time. In other words, the *PEF* which is a result of the pessimistic path cost, planned by the users of the network to avoid cost of reaching destinations α percent of the time, cause a planned vehicle-hour for the network of which only about 93 to 92 percent are actually required to traverse the network. The rest play the role of an insurance cost.

5.2 A preliminary result on a real network.

Recently, the network of the city of Shiraz, Iran, has been the subject of a study in which traffic counts have been carried on 29 links constituting a screen-line (called E) of this network, as well as 108 other selected links in this network. These counts are done for the morning peak of a working day in 2005. The city has an estimated population of about 1.3 million, and the network has about 1100 nodes and 1700 links, in this year. A comprehensive traffic assignment model was built about 5 years ago, which replicates the then observed link traffic volumes and path travel times, as well as several other statistics such as fuel consumption. This model which enjoys a *UE* flow routine, has been calibrated to replicate the newly collected count information mentioned above. The model has been revised to host a link-based *PEF* routine instead of its own *UEF* one, based on which the model was calibrated to best serve its purpose.

The new link cost function for *PEF* routine has been chosen to be of the following form for $\alpha = 0.98$:

$$\tilde{t}_{ij} = t_{ij} + K_{0.98} S_{ij} \approx t_{ij} + (2.0).\psi \left[\left(\frac{t_{ij}}{t_{ij}^{\circ}} \right) - 1 \right] \sqrt{t_{ij}}$$

where β in Equation 4 is assumed to be equal to 1.0, and where ψ is parameterized here to find the best value for the network of concern. Table 2 presents the results of the following regression analysis for various cases examined (as discussed below):

$$x_{ij}^{predicted} = \text{int} ercept + slope (x_{ij}^{observed})$$

The above regression shows how the estimated flows in the network in each case follow the respective observed ones. As may be seen in this table, for both 29 and 137 observation cases, $\psi = 0.1$ for $\beta = 1.0$ increases R^2 by a slight amount and decreases the standard deviation of the observations relative to the regression line. Moreover, for both cases, the position of the regression lines are improved at the same time (closer to zero intercept and/or closer to 1.0 slope). These results seem encouraging. Never-the-less, further evidences are required in this respect to make such observation concrete. (It is worth nothing that the range of variation for the value of R^2 for a calibrated model such as traffic assignment is around 2nd and 3rd decimal points).

Table 2. Results of regression analysis between model predicted and observed link volumes for the City of Shiraz.

ψ	Screen-line volumes (29 observations)				All link volumes (137 observations)				Remark
	Intercept	Slope	R^2	STD^{**}	Intercept	Slope	R^2	STD^{**}	
0.00	14.747	0.919	0.632	616.884	121.497	0.834	0.696	537.495	*UEF*
0.05	-10.167	0.922	0.643	604.023	113.864	0.839	0.706	528.353	
0.10*	-9.898	0.922	0.644	603.542	117.685	0.840	0.707	527.114	
0.20	-15.333	0.922	0.643	604.388	124.425	0.839	0.703	531.222	*PEF*
0.25	-16.179	0.925	0.640	610.971	124.637	0.843	0.701	536.705	
0.50	-56.731	0.978	0.615	681.097	123.368	0.862	0.660	604.002	
0.75	454.605	0.984	0.444	969.741	147.607	0.859	0.616	660.582	

* The better value.
** *STD* : standard deviation of observations relative to regression line.

5.3 A numerical example for PND

Figure 4 presents a small network with 5 nodes and 4 links. The volume delay-functions of the links, the O/D demand values, the standard deviations of link travel times, and other parameters of the problem are depicted in parts (a) and (b) of this figure. This figure shows that there are four candidate projects that may be chosen to be included in the network. Two of these projects are new links, and the other two are improvements in two existing links. Figure 4b. specifies the project characteristics (cost function, and cost of construction). All projects are of unit cost, and the budget levels are chosen to be 1, 2, or 3 units. Moreover, projects "c" and "d" are chosen to have higher capacity (lower congestion term coefficient, 1.0×10^{-7}, instead of 90×10^{-7} for projects "a" and "b"), but higher free flow travel times as compared with projects "a" and "b". The free flow travel times of projects "c" and "d" are parameterized to see the effect of this characteristic of these two projects on the results: δ in the respective travel time functions plays this role.
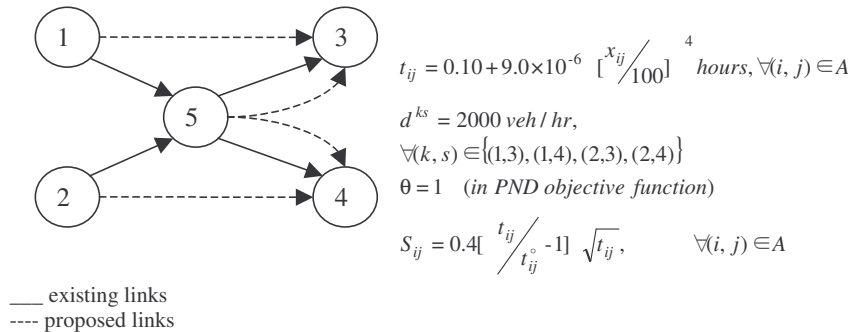It is not the intention of this paper to present an efficient algorithm to solve *PND* problem. Any suitable algorithm may be used for solving *CND* and *PND* problems, including an exhaustive enumeration procedure for this small problem in our case. Figure 4c. shows the results of solving *CND* and *PND* problems.
It can be seen in this figure that in some cases the results are not similar. In pondering the solutions of these two problems, one may note that when $B = 1$ the two problems have the same solutions: when δ is 50 or 60, project "c" is the solution to *CND* or *PND* problems, but when "c" becomes too long by higher δ (=70) project "a" becomes this common solution.

Increasing the budget level to 2 units would change the solutions of the two problems: For the lower value of δ (=50) projects "c" and "d" solve problem *PND*, while projects "a" and "d" solve problem *CND*. That is, in *PND* the project with higher capacity (though higher free flow travel time) is preferred to the one with lower values of these two terms. The same phenomenon happens for $B = 3$ and δ=60, in which project "d" replaces project "b" in the solution of problem *CND* to form the solution to problem *PND*. To elaborate further on this phenomenon, note that for this example problem one may write:

$$S_{ij} = 0.2 \, (t_{ij}/t_{ij}^\circ - 1) \sqrt{t_{ij}} = 0.2 \, \phi \, (x_{ij}/\phi_{ij})^4 \, [t_{ij}^\circ (1 + \phi \, (x_{ij}/\phi_{ij})^4)]^{0.5}$$

where ϕ is a constant, and ϕ_{ij} is the practical capacity of link (i, j). The above relationship shows that higher link capacity decreases link travel time variability (and thus increases its reliability). On the other hand, assuming equal average free flow link travel speeds, it follows that higher free flow travel times result from longer links. Thus, choice of projects "c" or "d", instead of projects "a" or "b", in the solution of *PND* means that, given every thing else the same (e.g., equal cost of project construction), *PND* tends to select long and high capacity projects "c" and "d", instead of "a" and "b" in *CND* (note that in the above discussion, because of the symmetry of network and demand, project "a" could replace project "b", and project "d" could replace project "c" in the solution set, and vice versa).

$$t_{ij} = 0.10 + 9.0 \times 10^{-6} \, [x_{ij}/100]^4 \quad hours, \, \forall (i, j) \in A$$

$$d^{ks} = 2000 \, veh / hr,$$
$$\forall (k, s) \in \{(1,3), (1,4), (2,3), (2,4)\}$$
$$\theta = 1 \quad (in \, PND \, objective \, function)$$

$$S_{ij} = 0.4 [\, t_{ij}/t_{ij}^\circ - 1] \, \sqrt{t_{ij}} \,, \qquad \forall (i, j) \in A$$

___ existing links
---- proposed links

4a. the existing network and proposed project links

$$t_{ij} = 0.10 + 9.0 \times 10^{-6} \, [x_{ij}/1000]^4 \quad , \quad (i, j) \in \{a = (5,3), \, b = (5,4)\}$$

$$t_{ij} = \delta \times 10^{-2} + 1.0 \times 10^{-7} \, [x_{ij}/1000]^4 \quad , \quad (i, j) \in \{c = (1,3), \, d = (2,4)\}$$

$$e_{ij} = 1, \forall (i, j) \in \{(5,3), (5,4), (1,3), (2,4)\}$$

4b. project specifications

Budget (B)	δ =50		δ =60		δ =70	
	CND	*PND*	*CND*	*PND*	*CND*	*PND*
1	c	c	c	c	a	a
2	a, d	c, d	a, d	a, d	a, b	a, b
3	a, c, d	a, c, d	a, b, c	a, c, d	a, b, c	a, b, c

4c. the optimal project sets for CND and PND

Figure 4. A numerical example for the PND problem

6. Summary and conclusions

Parkhurst et al. (1992) state based on a questionnaire–assisted study that users of urban transportation systems consider quality of service an important aspect of these systems. The uncertainty related to travel time has been the subject of complaints in most questionnaires. Data collected from urban street networks show that travel time in these streets is a stochastic variable, whose range of variation increases with the congestion level in them. Moreover, it is most important to be on time when making a trip for purposes such as work or school. This makes travelers to act conservatively in route choice, and choose routes and the starting time in such trips, which appropriately takes into account the uncertainty associated with the travel time.

Traffic volume (and hence, time) fluctuations affects travelers' route choice, and they choose routes which are short and with low travel time variation. This suggests due considerations of traffic fluctuations in problems related to street networks. This study has introduced a pessimistic equilibrium flow problem to describe the route choice process in a situation where travelers demand a high probability to be on time. This is the kind of situation found in morning peaks, where most travelers are destined to work (and in some cases, school). This notion reflects itself into the design of the transportation networks which are mainly based on a design O/D derived basically from morning peak periods. Thus, this paper proposed a pessimistic network design (PND) problem, which is based on a pessimistic equilibrium flow (PEF). These are the main contributions of this paper.

In a (PEF), travelers choose routes based on minimum average travel time plus its variability, such that in α percent of the times, destinations are reached on, or before, time . In a sense, (PEF) may be considered as a generalization of (UEF). An algorithm is proposed to solve this problem of flow incrementally. It has been demonstrated through some example networks that (PEF) is different from user equilibrium flow (UEF). This is true, particularly, when there is congestion in the network. An interesting consequence of a pessimistic route choice equilibrium is that the total travel time in the network (total vehicle-hr) for a PEF is sometimes lower than that for a user equilibrium flow (this, of-course depends on the relative magnitude of the two parts of the link cost function).

The proposed pessimistic network design problem pays due attention to the travel time fluctuations in the network. Traffic flow in this design problem is of PEF type. Application of this design concept on a small network revealed that PND tends to prefer long and high capacity projects rather than shorter and more prone to congestion ones. In other words, it chooses projects with lower potential for getting congested.

The followings are some avenues for further research: (a) specifying the degree of pessimism in route choice in morning peaks and possibly defining a mixed (i.e., UEF and PEF) problem; (b) specifying whether pessimism in route choice applies path-based (problem PEF) or link-based (problem $PEF1$); (c) estimating an appropriate value for the value of θ in the objective function of PND problem; (d) presentation of an efficient algorithm to solve PEF; (e) presentation of an efficient algorithm to solve PND; (f) studying the various extensions of the problem to include other travel choices or parameter and variable generalizations; and last but not least (g) analyzing the importance of the changes that would happen in decisions by considering PEF instead of UEF in real cases.

Acknowledgements

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Appendix A

Lemma 1. $\hat{t}_{\rho}$ is a strictly monotonic function of the flow vector.

Proof. Consider path $\rho \in \rho_{ks}$, for some $(k,s) \in P$. Let x_{ij} increase from x_{ij}' to x_{ij}'', for some $(i,j) \in \rho$. Then, $t_{ij}(x_{ij})$ increases from $t_{ij}' = t_{ij}(x_{ij}')$ to $t_{ij}'' = t_{ij}(x_{ij}'')$ by monotonicity of $t_{ij}(x_{ij})$.

From which one may conclude that $S_{ij}(t_{ij}) = \alpha\,(\frac{t_{ij}}{t_{ij}^{\circ}} - \beta)\sqrt{t_{ij}}$ increases from $S_{ij}' = S_{ij}(t_{ij}')$ to $S_{ij}'' = S_{ij}(t_{ij}'')$:

$$S_{ij}'' - S_{ij}' = \alpha\,(\frac{t_{ij}''}{t_{ij}^{\circ}} - \beta)\sqrt{t_{ij}''} - \alpha\,(\frac{t_{ij}'}{t_{ij}^{\circ}} - \beta)\sqrt{t_{ij}'} > \alpha\,(\frac{t_{ij}'}{t_{ij}^{\circ}} - \beta)\sqrt{t_{ij}''} - \alpha\,(\frac{t_{ij}'}{t_{ij}^{\circ}} - \beta)\sqrt{t_{ij}'} =$$

$$\alpha\,(\frac{t_{ij}'}{t_{ij}^{\circ}} - \beta)(\sqrt{t_{ij}''} - \sqrt{t_{ij}'}) = \alpha\,(\frac{t_{ij}'}{t_{ij}^{\circ}} - \beta)(\frac{t_{ij}'' - t_{ij}'}{\sqrt{t_{ij}''} - \sqrt{t_{ij}'}}) > 0 \quad, for\ \beta < 1 \quad.$$

Increases in S_{ij} would increase $S^2 = \sum_{(i,j) \in \rho} S_{ij}^2$ from S'^2 to S''^2, which would in turn increase

$S_{\rho} = \sqrt{\sum_{(i,j) \in \rho} S_{ij}^2}$ from S_{ρ}' to S_{ρ}'':

$$S_{\rho}'' - S_{\rho}' = \sqrt{S_{\rho}''^2} - \sqrt{S_{\rho}'^2} = \frac{S''^2 - S'^2}{\sqrt{S''^2} + \sqrt{S'^2}} > 0$$

This would in turn increase $\hat{t}_{\rho}$ from $\hat{t}_{\rho}'$ to $\hat{t}_{\rho}''$:

$$\hat{t}_{\rho}'' - \hat{t}_{\rho}' = (\sum_{(i,j) \in \rho} t_{ij}'' + K_{\alpha} S_{\rho}'') - (\sum_{(i,j) \in \rho} t_{ij}' + K_{\alpha} S_{\rho}') = (\sum_{(i,j) \in \rho} t_{ij}'' - \sum_{(i,j) \in \rho} t_{ij}') + K_{\alpha}(S_{\rho}'' - S_{\rho}') > 0.$$

Thus, $\hat{t}_{\rho}$ is a strictly monotonic function of x. □

Lemma 2. Let $t_{ij}(x_{ij})$ be the link (i,j) travel time as a function of flow x_{ij} only, strictly convex, and twice differentiable defined for $x_{ij \geq 0}$. Let S_{ij} denote the standard deviation of

the travel time in link (i, j) as defined in Equation (4) of the paper. Then, $\tilde{t}_{ij} = t_{ij} + K_\alpha S_{ij}$ is a strictly convex function of x_{ij}.

Proof. Let us delete the indices i and j from t and S for ease of presentation. For $S = \psi(\dfrac{t}{t^\circ} - \beta)\sqrt{t}$, it may be shown that:

$$\frac{dS}{dx} = \frac{dS}{dt} \times \frac{dt}{dx} = \psi[\frac{3\sqrt{t}}{2t^\circ} - \frac{\beta}{2\sqrt{t}}]\frac{dt}{dx};$$

and:

$$\frac{d^2 S}{dx^2} = \frac{d}{dx}(\frac{dS}{dx}) = \frac{d}{dt}(\frac{dS}{dx}).\frac{dt}{dx} = \psi(\frac{3t + \beta t^\circ}{4t^\circ t\sqrt{t}})(\frac{dt}{dx})^2 + (\frac{3t - \beta t^\circ}{2t^\circ \sqrt{t}})(\frac{d^2 t}{dx^2})$$

Since $S \geqslant 0$, it follows that $\dfrac{t}{t^\circ} - \beta \geqslant 0$, or $t \geqslant \beta t^\circ$. And, since $t(x)$ is nonnegative and strictly convex $\dfrac{dt}{dx} \geqslant 0$ and $\dfrac{d^2 t(x)}{dx^2} > 0$. Thus, $\dfrac{d^2 S}{dx^2} > 0$, and hence S is strictly convex. $\tilde{t}$ is the sum of two strictly convex functions, and hence is strictly convex.□

Lemma 3. Given any flow pattern x for a given network, $\hat{t}_\rho^{ks} < \tilde{t}_\rho^{ks}$, for any $\rho \in \rho^{ks}$, and any $(k, s) \in P$.

Proof. For any $\rho \in \rho^{ks}$, and $(k, s) \in P$, one may write:

$$\hat{t}_\rho^{ks} = t_\rho^{ks} + K_\alpha S_\rho^{ks} = \sum_{(i,j)\in\rho} t_{ij} + K_\alpha \sqrt{\sum_{(i,j)\in\rho} S_{ij}^2} < \sum_{(i,j)\in\rho} t_{ij} + K_\alpha \sum_{(i,j)\in\rho} S_{ij} = \sum_{(i,j)\in\rho}(t_{ij} + K_\alpha S_{ij}) = \tilde{t}_\rho^{ks}$$

where the inequality sign has been written by noting that for a set of a_i's, $a_i > 0$ for all i, one has:

$$\sqrt{\sum_i a_i^2} < \sum_i \sqrt{a_i^2} = \sum_i a_i.□$$