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A new way of determining distance decay parameters in spatial interaction models with application to job accessibility analysis in Sweden

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In this paper we explore and compare various techniques for the calculation of distance decay parameters which are estimated using statistical methods with half-life decay parameters which are derived mathematically. Half-life models appear to be a valid alternative to traditional spatial interaction models, especially in the presence of spatially highly disaggregate data. Our results indicate that Half-life models are more accurate for the construction of decay parameters than are unconstrained spatial interaction models in 'medium' sized datasets but not as accurate as doubly-constrained models. However, using highly detailed and disaggregate datasets Half-life models may be viable alternatives to doubly-constrained spatial interaction models as the latter will be difficult to estimate when the number of origins and destinations increase. In addition, Half-life models rise in accuracy with increasing degrees of disaggregation due to reductions of systematic errors between observed individual level commuting distance and modelled distances between origins and destinations.

In sum, our findings are as follows. First, since unconstrained and doubly-constrained spatial interaction models become increasingly difficult to estimate and/or less accurate to use compared to Half-life models as the spatial disaggregation increases choice of decay parameter estimation model should be considered in relation to level of disaggregation. Secondly, Half-life models are not affected by the systematic errors observed in the statistically derived models. Finally, using Half-life models for the estimation of decay parameters is simple which may make it easy to employ among practitioners lacking skills or computer means for the estimation of more complex statistically derived models.

Keywords: spatial interaction models, half-life, distance decay, potential accessibility, commuting in Sweden

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1. Introduction

Most analyses of flows of people to and from jobs, services or similar are dependent on the quality of the distance decay parameters that are estimated for the spatial interaction analyses. This means, for example, that estimates of accessibility to jobs, services, recreational facilities or other amenities may vary significantly not only due to spatial clustering or relative proximity to what is being studied but also due to differences in how the friction of distance is modelled in the analyses.

Accessibility is commonly estimated using SIMs (Spatial Interaction Models) in which observed distances and flows of people between origins and destinations are used as input into modelling. Commonly, accessibility is estimated using SIM's where the flow of people over various distances is determined by the mass of the attractions at the destinations and a distance deterring function (Hansen, 1959). Determining decay rates for distance deterring functions in SIMS is usually conducted with regressions where distances between origins and destinations are regressed against the observed flow of people between all origins and destinations. More accurate but also more computational demanding models for the estimation of decay parameters (for example singly and doubly-constrained models) are computed using iterative statistical methods (Wilson 1970). Since computers are getting increasingly fast over time – these more complex iterative models are becoming less and less demanding to execute.

There are however two issues that might force researchers to look at completely different distance decay parameter estimation models. First, in many cases there is an abundance of data describing number of jobs and homes in local regional statistics for many countries around the globe. However, flow-data describing the flow of commuters between and within regions are much more difficult to retrieve and in many cases there is no collection of these data at all. Travelling surveys can in many cases be used to depict general local commuting behaviours – though in the absence of origin-to-destination flows, traditional models cannot be employed. In situations like these – alternative methods for the estimation of distance decay parameters can be useful. Secondly, in an increasing number of regions and countries, individual level or spatially very disaggregate statistics are available. However, with increasing disaggregation comes increasing difficulties with the iterative calculation of constrained decay parameters. This partly because the number of potential interactions quickly increases as the number of studied units is growing, making computations very computer demanding, partly because at some point in the disaggregation of data, a majority or even all of the observed flows between origins and destinations become unique. In these situations the balancing factors used to calculate iteratively based constrained parameters will be impossible or meaningless to compute. Under these circumstances alternatively specified models for the estimation of decay parameters may be useful. Obviously, in the presence of disaggregated data, with statistics regarding available modes of transportation and/or statistics that can be used to estimate choice probabilities for spatial interaction, Multinomial Logit (MNL) models can be adopted too. MNL models display strong economic theoretical roots and have long been used in transport planning (see for instance, McFadden 1974; Train 1978; Anas 1983). However, individual level statistics of the kind needed for MNL modelling is often difficult to obtain.

In this paper, and in the mentioned accessibility analysis, we set out to test how well new methods for estimation of distance decay work when applied in two widely used SIMs using common specifications of distance decay. First we discuss the theoretical and methodological basis for spatial interaction analysis and for the estimation of distance decay parameters in particular. Three families of models for the estimation of decay parameter are discussed: unconstrained, doubly-constrained and half-life models (Section 2). In Section 3, two datasets used in our empirical application (compiled for studies of job accessibility in Sweden) are given a thorough presentation. In Section 4, results from the comparative studies are presented, with a view on the emerging accessibility patterns. Finally in Section 5, general conclusions about under what circumstances which kinds of distance decay parameter models may be applicable in accessibility analyses are drawn.

2. Modelling Distance Decay and Spatial Interaction

When employing potential models for the estimation of accessibility, not only the quality and disaggregation of data describing flows, attractions at destination and situation at place of origin affects the outcome. A large part of the estimated accessibility can be attributable to the choice of interaction model and to choice of decay function. In sub-section 2.1 the decay functions employed in this paper are described and in sub-sections 2.2, 2.3 and 2.4 two types of SIMs and three methods for the estimation/calculation of distance decay models are presented.

2.1 Distance Decay Models

Spatial interaction between locations is determined by a multitude of factors including spatial organisation of home and work, infrastructure and utility for commuter to mention a few. This means that commuting distances/times often are non-linear indicating that choice of SIM is important for the modelling outcome (Johansson et al. 2003).The choice of the SIM clearly affects the 'best' *ß*-value to be introduced, and thus its outcome. Besides choosing SIM, choosing type of decay function is crucial. Discussions in this respect have been provided recently, with application to the German commuting flows (Reggiani 2012; Reggiani et al., 2011). In this particular German context, five decay functions have been adopted and tested. These decay equations are:

a) the exponential-decay function:

$$f(d_{ij}) = e^{-\beta d_{ij}} , \tag{1}$$

b) the power-decay function:

$$f(d_{ij}) = d_{ij}^{-\beta} , \tag{2}$$

c) the exponential-normal decay function:

$$f(d_{ij}) = e^{-\beta d_{ij}^2} , \tag{3}$$

d) the exponential-square-root decay function:

$$f(d_{ij}) = e^{-\beta \sqrt{d_{ij}}} , \tag{4}$$

e) the log-normal-decay function:

$$f(d_{ij}) = e^{-\beta \ln(d_{ij})^2} , \tag{5}$$

where the coefficient β represents the distance-sensitivity parameters.

Discussion on the different properties of these functions have already been provided, among others, in De Montis et al. (2011), De Vries et al. (2009), Reggiani et al. (2011), Willigers et al. (2007), by essentially discussing the potential of the exponential decay function vs the power decay functions (Eqs (1) and (2)), on the basis of the fundamental works of Fotheringham and O'Kelly (1989) and Wilson (1981). A subsequent work by Östh et al. (2014) applies Eqs. (1) and (2) to job accessibility on municipality level in Sweden.

In the present analysis, three different methods are used to estimate the decay parameters in this paper; two of the methods can be considered as common, while the third to large extent is new in

SIM. The first method makes use of an unconstrained approach in which decay is estimated using regressions. The second method considered is the doubly-constrained approach in-which the decay parameters are estimated regressive and iterative. Specifications of unconstrained and doubly-constrained SIM are found in sections 2.1.1 and 2.2.2.

A final step is to compare, statistically as well as visually using maps of accessibility patterns, the above listed more common decay function parameters with those emerging from the half-life models. All three models are presented in the subsequent three sections.

2.1.1 The Unconstrained Spatial Interaction Model

TSIM is a static model designed to predict the magnitudes of spatial mobility, i.e. the processes or spatial flows emerging as result of given spatial configurations. Consequently, SIMs represent flows of people, commodities, capital, information, etc., between some origin i to some destination j. The SIM gained a lot of popularity in the past for their usefulness in studying mobility and is still considered relevant for exploring the cohesion and dispersion of activities in spatial systems (Östh et al., 2014; Reggiani, 2012; 2014).

In Östh et al. (2014) SIMs have been widely described, on the basis of the fundamental work of Wilson (1970; 1981); subsequent work has provided a strong theoretical foundation linked to entropy theory, and thus to the utility maximising approach, and whose work came to bridge methods in transport analysis with regional economics into a common framework (Anas, 1983; Mattsson 1984; Nijkamp and Reggiani, 1992; O'Kelly, 2010). From here, SIMs have been interpreted as aggregate models of human behaviour. Three main forms of SIM exist: a) the unconstrained SIM; b) singly-constrained SIM and c) the doubly-constrained SIM. The general form of the unconstrained SIM – which is directly linked to the analogy with Newton's law of gravity – can be specified as below:

$$T_{ij} = K \cdot O_i \cdot D_j \cdot f(\beta, d_{ij}) \quad (6)$$

Where T_{ij} represent the number of flows between the origin i and the destination j. These interaction flows are a function of the outflows O_i and of the inflows D_j, as well as of the distance decay function $f(ß, d_{ij})$; d_{ij} represents the generalized cost, time or distance between i and j, and the parameter K is a scaling factor, which results from the calibration on real data (to facilitate comparison between models no K parameter is used in this paper). The decay parameter $ß$ determines, on an aggregate level, the travelling behaviour in the studied population. The $ß$-value emerging from the calibration of Eq. (6) will be the core element in our empirical analysis of unconstrained SIMs. Two decay functions are commonly used in unconstrained SIMs, exponential-decay function and power-decay function (Eqs. 1 & 2). These decay functions are commonly calibrated using regression techniques where the dependent variable y is expressed as $\ln(T_{ij}/(D_jO_i))$, i.e. ln(observed flow between zone $_i$ and $_j$ / (number of jobs in $_j$ *number of workers residing in $_i$)) and where the independent variable x represents distance d_{ij} between zone $_i$ and $_j$, ($\ln d_{ij}$ in the power model).

2.1.2. The Doubly-Constrained Spatial Interaction Model

In contrast to the unconstrained SIM, the doubly-constrained SIM considers interaction between origin i and destination j by incorporating restrains on both the supply and demand side[4]. The general form of the doubly-constrained SIM is the following:

$$T_{ij} = A_i B_j O_i D_j f(ß, d_{ij}), \qquad i = 1, \ldots, I; \qquad j = 1, \ldots, J \quad (7)$$

[4] Also singly-constrained SIMs, which balances either the supply or demand, exists. However, given their specificity, the singly-constrained SIMs are not analyzed in our experiments, aiming to extract the optimal decay parameters to be used in the accessibility functions

where the variables are the same variables as in Eq. (6). The main difference here concerns the emergence of the balancing factors A_i and B_j, in substitution to the parameter K in Eq. (6). In particular, A_i and B_j reads as follow:

$$A_i = 1 \;/\; \sum_j B_j D_j \; f(ß, d_{ij}); \qquad B_j = 1 \;/\; \sum_i A_i O_i \; f(ß, d_{ij}), \tag{8}$$

Since they come out from the related additive conditions:

$$\sum_j T_{ij} = O_i\,; \qquad \sum_i T_{ij} = D_j \tag{9}$$

Wilson (1970; 1981) provides the form of the impedance function $f(ß, d_{ij})$, by considering the following constraint on the total distance d^* (or cost), in addition to the constraints expressed in Eq. (9):

$$\sum_{ij} d_{ij} T_{ij} = d^* \tag{10}$$

Thanks to Wilson's entropy approach, the doubly-constrained SIM, expressed in Eq. (7), can be interpreted in a macro-behavioural context, in terms of a generalised cost function for spatial interaction behaviour (Nijkamp, 1975), as well as in a micro-economic context, given its formal equivalence with the family of logit models (Reggiani, 2012). This macro-micro behavioural framework provides an economic perspective to the doubly-constrained SIM (7). Also in doubly-constrained SIM, two decay functions are *commonly* used, i.e. exponential-decay function and power-decay function (Eqs. 1 & 2). The iterative procedures employed for the calibration of the doubly-constrained model are often complex and time-consuming[5].

2.1.3. The Half-Life Model

Mathematically derived half-life models (HLMs) are commonly used to express decay of substances in physics and for similar issues in other scientific fields but relatively uncommon in transport studies, planning, geography and spatial economics.

The general form of the half-life SIM is identical to the unconstrained SIM presented above. The difference between the two types of models is how β is calibrated.

$$T_{ij} = K \cdot O_i \cdot D_j \cdot f(\beta, d_{ij}), \tag{11}$$

In spatial analysis, decay of potential interaction between locations is commonly determined by the distance, cost or time between locations. This means that we theoretically should be able to estimate the decay of potential interaction between locations if we know the distance, cost or time between the locations. Statistically, decay of spatial interaction is estimated using the techniques described in the earlier sub-chapters, but in order to determine decay parameters mathematically, observations need to be handled differently. To exemplify, if we utilize data from travelling-surveys, GPS-recorders or registers of residential locations and workplaces (as in this study) we can derive both mean and the median commuting distance in a given population. While the commonly used statistical models aim to reduce the overall deviation from the mean when estimating the decay parameter, HLMs depart from the median value. The reason is that the median commuted distance (or time or cost for that matter) always occur at a distance where half of the population commute longer and half of the population commute shorter, whilst the mean commuting distance (usually) have different and varying shares of the population on either side of the mean value. By departing from the median commuting distance we can state that for any

[5] The iterative search for successively better approximations of A_i and B_j values are conducted using a Newton–Raphson method. In the Appendix calibration statistics are described.

commuter, the probability of being employed equals 0.5 at the observed median distance. Following this, if we employ a decay function to describe the probability of being able to hold a job at various distances, the probability-value will decay from one at no distance towards almost zero at far, far away. Since half of the population commuted to a job on a distance between zero to median commuting distance, we can assume that the sum of job-probabilities/accessibilities over distance ought to be half of the sum of all job-probability/accessibility at the median distance.

Being able to associate zero to median commuting distances to one half of the population and median to maximum commuting distances to the other half of the population means that the median distance commuted intersects where half of the AUC (Area Under the Curve) of an integral function describes access to jobs[6]. If the distance decay pattern of spatial interaction in a work commuting dataset decays in a way that is similar to the decay patterns in any of the decay functions listed above (exponential, exponential normal, exponential square root or log-normal), high correlations between observed interaction and estimated interaction should be observable.

We have only come across two papers in which a HLM specification of exponential distance decay is being used in spatial analysis (O'Kelly & Horner, 2003; Östh et al., 2014). In this paper we expand the use of HLMs to encompass several decay functions. Because HLMs are relatively uncommon in this field a somewhat more lengthy discussion on their mathematical basis, as well their potentials and limitations are needed. As mentioned above, half-life parameters are derived using median commuting distances. In highly aggregate datasets this will lead to relatively large systematic errors. This since the deviation between the observed median distance and distances between big, aggregate spatial units will be relatively large. If for example spatial interaction between the 8 NUTS-2 regions in Sweden is under analysis – the deviation between observed population median commuting distance and the distances used in a cost matrix for NUTS-2 will be very large. In analogy, with increasing disaggregation, the deviation between median distance and distances between units will decrease reducing the systematic error. This type of systematic error will be eliminated once spatially non-aggregated data is being used.

In the subsequent text the mathematical basis for the calculation of half-life *ß*-values for exponential decay, exponential-normal decay, exponential square-root and the log-normal decay function is presented. HLM parameters for power decay functions cannot be calculated mathematically. This because the power function is asymptotic on the x-axis making calculations of AUC unachievable. For the exponential function the integral and the solution for finding the decay parameter is described in the text – solution for the remaining three models are moved to the appendix. To facilitate the calculation of half-life decay parameters a website has been created from which parameters for the four decay functions can be estimated with no other requirements than an idea about the median distance and a web-browser supporting JavaScript[7].

2.1.3.1. The Adopted Half-Life Decay Functions

Perceiving of distance decay as an integral function, the total AUC (Area Under the Curve) can be interpreted as the sum of access to one object over a span of distances. This total area can, for the exponential function, be formulated mathematically as an integral (Eq. (12)):

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[6] The exponential half-life model has certain properties that make estimation of HLM relatively straightforward. For the exponential decay function, the median commuting distance can be used not only to separate the population in two equally sized parts (commuting longer and shorter respectively) but is also a distance where the probability for commuting equals ½. For the other models, these two properties do not coincide. This means that the distance in the X-axis intersecting with ½ of AUC ≠ ½ probability to commute longer or shorter. See Appendix A2 for a graphical illustration of relationships.

[7] Link to website: http://equipop.kultgeog.uu.se/Decay/untitled.html

$$\int_0^{\infty} e^{-\beta x} dx = 1/\beta \tag{12}$$

Where ∫ represents the integrated area between distance zero (0) and eternity ∞, $e^{\beta x}$ represents the exponential function and *dx* represents an infinitesimal change in x. Since the distance to 'half-life' of commuting coincides with half of the AUC, the formulation of the integral for half-life and half AUC can be formulated as in Eq. (13) or (14):

$$\int_0^{m} e^{-\beta x} dx = 0.5/\beta \tag{13}$$

$$0.5 = \beta \int_0^{m} e^{\beta x} dx = 1 - e^{-\beta m} \tag{14}$$

The differences between Eq. (12) and Eqs. (13)-(14) consist of changes in the span of distance from zero to *m*, as well as a reduction of the integrated area from 1 to 0.5. *m* is in this paper represented by the median commuting distance in Sweden in 2010 (~6010m). The remaining unknown value is the parameter (*β*) which can be determined rewriting Eq. (14) as in the Eq. (15) below:

$$0.5 = e^{-\beta m} \tag{15}$$

Taking natural logs (ln):

$$\ln(0.5) = -\beta m \tag{16}$$

And finally solving for *β*, we obtain:

$$\beta = -\frac{\ln(0.5)}{m} \tag{17}$$

The decay parameter calculated from (17) is the HL decay model embedded into the exponential decay function (1).

For the remaining three functions (exponential-normal, exponential square-root and log-normal) only the solutions are presented below. Details can be found in the Appendix. The same logic as for the exponential function applies to these functions as well. The mathematical solution for the calculation of a decay parameter to be used in the exponential Normal function, Eq. (3), is as expressed in Eq. (18):

$$\beta = \left(\frac{erf^{-1}(0.5)}{m} \right)^2 \tag{18}$$

Where $erf^{-1}(0.5)$ represents the inverted error function at half (0.5) of the integrated value. At 0.5 this value equals approximately 0.47693628.

For the square-root function, Eq. (4), the solution for obtaining *β* is expressed in Eq. (19):

$$\beta \approx \frac{1.67835}{\sqrt{m}} \tag{19}$$

The decay parameter function (19) is one of the two solutions emerging from Eq.(4). The alternative solution is visible in Eq. (A11), in the Appendix. However, since only function (19) is decaying with increasing distance this is the only one to be considered.

Due to the ± sign in the equation, the log-normal decay parameter equation (20) has two solutions. These two are from now on described as log-normal (plus) and log-normal (minus). The two parameters are used in the log-Normal decay function, Eq. (5).

$$\beta = \frac{\left(erf^{-1}(-0.5)\right)^2 \pm \sqrt{\left(erf^{-1}(-0.5)\right)^4 + 2\left(erf^{-1}(-0.5)\right)^2 \ln(m)} + \ln(m)}{2(\ln(m))^2} \tag{20}$$

In the Appendix formulations and solutions are presented more thoroughly. In the framework of our empirical application to the commuting flows in Sweden, we will use Eq. (17) in the exponential decay function (1), Eq. (18) in the exponential-normal decay function (3), Eq. (19) in the square-root decay function (4) and finally Eq. (20) in the log-normal decay function (5). We will then compare the emerging results with those derived from the conventional SIMs (illustrated in Sections 2.1 & 2.2). The findings of this comparative analysis will be illustrated in Section 4.

3. Data and case studies

Two datasets are used and analysed in this paper, the first dataset describes Swedish commuting on a municipality level in year 2010 while the second dataset makes use of flows of commuters to and from 5km x 5km gridded units. Using datasets with different scales offers a possibility to test if half-life derived decay parameters behaves similar or different to parameters derived using traditional computational methods at different scales.

Data for both datasets were drawn from the Uppsala University based PLACE-database. The database contains socio-economic, employment-related and demographic variables as well as residential and workplace coordinates of all Sweden-resident individuals between 1990 and 2010[8]. Both the municipality and the 5km grid datasets contain four variables. These variables are: origin (place identifier), destination (place identifier), commuting distance (between origin and destination) and flow (count of commuters). The distance variable was constructed using individual-level data on coordinates of work and home for the calculation of Cartesian distances. The calculated individual distances were aggregated to municipality and to 5km levels so that the median Cartesian distance commuted between any origin and destination could be retrieved and used in our models. For the HLM the median distance is required. Using the Cartesian distance for all individuals' recorded home-to-workplace distances a median commuting distance of 6010 meters was recorded for Sweden 2010. Since the median distance is based on individual-level data, the median distance and the resulting decay parameters are valid in both of the datasets tested in this paper. Using Cartesian distance between home and work to represent the commuting distances can be criticized for not taking the network distance into account. Alternative distance specifications would make use of observed cost for interaction or time spent commuting. However, in the absence of commuting data on levels allowing for analysis also on 5km x 5km, Cartesian distance must be considered as best available alternative. It should be noted that unconstrained, doubly-constrained and half-life models can be executed also using alternative distance specifications where available.

The datasets have been compiled so that all possible flows between places of origin and destinations are represented by cases. The first case-study, referred in the subsequent Section 5.1 as 'small to midsized dataset', is represented by the Swedish municipality dataset, which comprises 290 municipalities * 290 municipalities = 84 100 cases. The second case-study, referred in the subsequent Section 5.2 as 'large dataset', is represented by the 5km x 5km unit dataset, which comprises 12 079 grid units * 12 079 grid units = 145 902 241 cases. In reality, less than half

[8] Individuals residing in Sweden during the last of December each year are recorded in the database.

of the municipality based origin-to-destination flows are occupied with actual flows. Flows between 5km units are, in relation to the total count, even scarcer. Missing flows between origins and destinations are replaced with zero.

4. Results

In our empirical application, the unconstrained, doubly-constrained and HLMs are tested in terms of how well they estimate flows of commuters between locations in small to midsized datasets and under what circumstances they may and may not be used for the analysis of accessibility. Two tests are conducted in order to review the usefulness of the employed decay parameters. In the first test the overall deviation between observed flows and estimated flows are measured using RMSE (Root Mean Square Error) [9]. The greater the deviation from the observed flows, the greater the RMSE value will be, indicating that the estimation under- or overshoots in flow prediction. However, since RMSE doesn't take the model fit into account, a second test of how well the estimates correlate with observed values will be conducted using Pearson correlation analysis. Knowing the model fit is useful in studies where the relative interaction or accessibility is of interest (in the appendix figure A1, test-differences between RMSE and correlations are illustrated).

4.1. Small to Midsized Datasets
The results from the tests applied to the municipality dataset are shown in Table 1. In the top row the decay parameter values are presented. Since different statistical and mathematical models were used for their calibration it is interesting but not surprising that their values vary also when they have been calibrated for the same SIM (as in the case of the exponential SIM where three decay parameters are presented). The focus for comparison is not the parameter-value per se but rather how well parameter and SIM produce credible and useful estimates. Analyses of how big the RMSE value is reveals that doubly-constrained models and parameters generated considerably lower RMSE compared to unconstrained and Half-life models and parameters. It is noteworthy that the RMSE for the unconstrained exponential model is very poor compared to all others. This indicates that the deviation between predicted flows and observed flows is considerable. The correlation tests were conducted to see to what extent the predicted flow of commuters correlated to the observed flow of commuters. The correlation results, displayed in the bottom row of Table 1, reveal that both of the doubly-constrained models are doing exceptionally good jobs in estimating flows. Remaining models, with the exception of the unconstrained exponential model, render similar correlation values of which the best correlation is recorded for the half-life log-normal (plus) model. That doubly-constrained SIMs render the best results is not surprising since the models cater for competition for job opportunities but that the statistically derived unconstrained models did similar or worse compared to the HLMs in terms of correlation values must be considered as an interesting finding. In sum, the results suggest that small to medium sized datasets benefit from using doubly-constrained decay parameters. The results also raise concerns regarding the use of the unconstrained exponential model since neither RMSE nor correlation coefficients render results that are close to the others in terms of test-results.

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[9] RMSE = $\sqrt{\sum_{i,j}(\hat{y}_i - y_i)^2}$ aggregate the errors in predictions.

Table 1 Distance decay parameters used in the municipality dataset

	UC Exponential	DC Exponential	UC Power	DC Power	HLM Exponential	HLM Exponential Normal	HLM Exponential Square-root	HLM Log-Normal (Minus)	HLM Log-Normal (plus)
Param*	.0000036	.0000167	1.373523	1.883556	. 0001153	6.3E-09#	. 0216493	. 0457406	. 0721908
RMSE	6 282 504	121 504	405 913	72 830	761 820	1 225 238	400 762	458 589	487 988
Corr.	.200**	.980**	.620**	.996**	.594**	.579**	.600**	.557**	.631**

* Table 1 show distance decay parameters used in the municipality dataset (row one); RMSE values in row two and Pearson correlation coefficients in row three. All decay parameters are estimated for distances measured in meters. UC represents UnConstrained, DC represents Doubly-constrained and HLM represents Half-Life Models. ** indicates that correlations are significant on 99.9% level, n = 84 100 (290 x 290, municipalities). # HLM Exponential Normal parameter value is too small to be shown in table. The derived value equals: 0,000000006297552

4.2. Large Datasets

For the second dataset the situation is relatively different. In this dataset the spatial interaction is estimated for 12 079 different 5km units being populated with either jobs, workers or both jobs and workers. A full matrix comprising of 145 902 241 rows (12 079 x 12 079 units) has been used for estimations of distance decay parameters and for the estimation of interaction. The half-life derived distance decay parameters need not to be recalculated since the values are valid at any spatial scale (median commuting distance of ~ 6010m is used on all scales) but the unconstrained beta values need to be re-estimated using the regressions specified in section 2.1. Estimation of unconstrained and half-life spatial interaction estimates turns out to be relatively simple and quick also in datasets of this size. However, the sheer number of units turns out to be far too big for the estimation of doubly-constrained distance decay parameters and interaction[10].

Compared to the municipality dataset RMSE is becoming worse for the unconstrained exponential SIM estimates in the 5km-dataset, whilst RMSE in improving for the unconstrained power SIM. However, correlating unconstrained exponential and unconstrained power estimates to observed flows not only shows that correlation coefficients decrease in comparison to corresponding coefficients in the municipality dataset, the correlation coefficients are also considerably lower than the half-life coefficients. For the half-life models correlation coefficients increase in the 5km dataset compared to the municipality dataset. With the exception for the exponential HLM, the RMSE test values are improving for all half-life estimates in the 5km dataset.

That correlation values increase for HLMs is likely partially a consequence of a reduction in the systematic errors, i.e. the deviation between the population median commuting distance used to determine decay parameter and the distances between and within 5km units used in SIMs is reduced compared to the municipality dataset.

10 Using a 26gb ram and double quad-core processers was insufficient to estimate doubly constrained accessibility for datasets reduced to a quarter of the size of the 5km-dataset (we did not create smaller dataset – so we are uncertain of the exact dataset-size-threshold which probably is considerably smaller on this computer).

Table 2 Distance decay parameters used in the large 5km-dataset

	UC Exponential	DC Exponential	UC Power	DC Power	HLM Exponential	HLM Exponential Normal	HLM Exponential Square-root	HLM Log-Normal (Minus)	HLM Log-Normal (plus)
Param*	.0000102	Na	1.5285710	Na	.0001153	6.3E-09#	.02164934	.0457406	.0721908
RMSE	8 890 522	Na	128 053	Na	909 126	1 365 067	343 082	107 978	115 885
Corr.	.158**	Na	.214**	Na	.699**	.655**	.735**	.588**	.737**

* Table 2 show distance decay parameters used in the 5km x 5km dataset (row one); RMSE values in row two and Pearson correlation coefficients in row three. All decay parameters are estimated for distances measured in meters. UC represents UnConstrained, DC represents Doubly-constrained and HLM represents Half-Life Models. ** indicates that correlations are significant on 99.9% level, n = 145 902 241 (12079 * 12079, 5km units). NA indicates that results are not available. # HLM Exponential Normal parameter value is too small to be shown in table. The derived value equals: 0.000000006297552

It is obvious that some of the models listed in table 2 perform better than others – however, the RMSE and correlation results are dependent on the spatial configuration of opportunities and the nature of supply and demand in Sweden. This means that if what is being studied (nature of) is migration on one extreme or friendship between kids in a neighbourhood on the other – which model that correlates best with observed flows may very well change. In addition, studies of commuting patterns in Sweden are to an unknown extent driven by the spatial organisation of society. Similar models in other countries may for the same reason lead to different results. A good way of understanding how the different models depict spatial interaction is to map the result[11]. However, since the flows between all origins to destinations contain too much information the spatial interaction estimates are aggregated so that each 5km unit holds the sum of potential flow of commuters. By aggregating the flows we end up with Hansen (1959) type of potential accessibility where the local potential accessibility (Acc_i) can be expressed as $Acc_i = \sum_j D_j f(\beta d_{ij})$. The related results are discussed in the next Section.

4.3. Mapping Accessibility

The last step of our analysis is the study of accessibility in Sweden, on the basis of the different decay parameters emerging from the various models considered. For this analysis we will consider the more detailed spatial unit (case study of large data set).

In lower right part of figure 1 the HLM exponential accessibility is illustrated using quintiles (low accessibility = blue, high accessibility = red). However, to enhance the model specific spatial behaviours the modelling output is normalized using the observed number of commuters at every location (potential accessibility over observed count of commuters, i.e. Acc_i / O_i). This way it is model specifics rather than spatial variation in accessibility that is being displayed. It is important to note that though potential accessibility values vary significantly between models, the size of accessibility values is not of interest for our methodological purpose. What matters is whether and/or how output varies systematically in response to magnitude of concentration of jobs, shape of studied area (Sweden) and proximity to borders. This is also why the normalized

[11] The municipality dataset is not mapped because the varying sizes and shapes of the municipalities make it difficult to display the model specific behaviors.

output is illustrated on the same scale, using 10 quintiles to differentiate between areas of 'under- and overshoot'.

The normalized outputs clearly indicate that there are both distinctive similarities and differences between models in how the spatial distribution of accessibility is displayed. All normalized output render greater potential accessibility values in the areas between the three major metropolitan areas of Sweden (see red core area in the mid-south of Sweden). A key reason for this is that the red areas have the overall shortest distances to all jobs in Sweden. HLM exponential, UC Exponential and HLM Log-Normal (minus) are very similar-looking with patterns showing 'overshoot' of accessibility in the southern parts and 'undershot' in the northern parts. Deviation in accessibility patterns seem to happen on a national level. The HLM exponential normal in particular but also the HLM exponential square root concentrates the overshoot to the southern inland areas while coastal areas and remote areas render low values. It is obvious that especially the exponential normal model is distance sensitive, 5km units outside urban areas almost immediately undershoots and borders and coasts are 'incapable' of getting high values since their surrounding search areas are spatially restricted. The UC power and HLM Log-Normal (plus) models split 'over- and undershoot' on an urban and a rural level. In the UC power model output, rural areas overshoot and urban areas undershoot more than the HLM Log-Normal (plus) model.

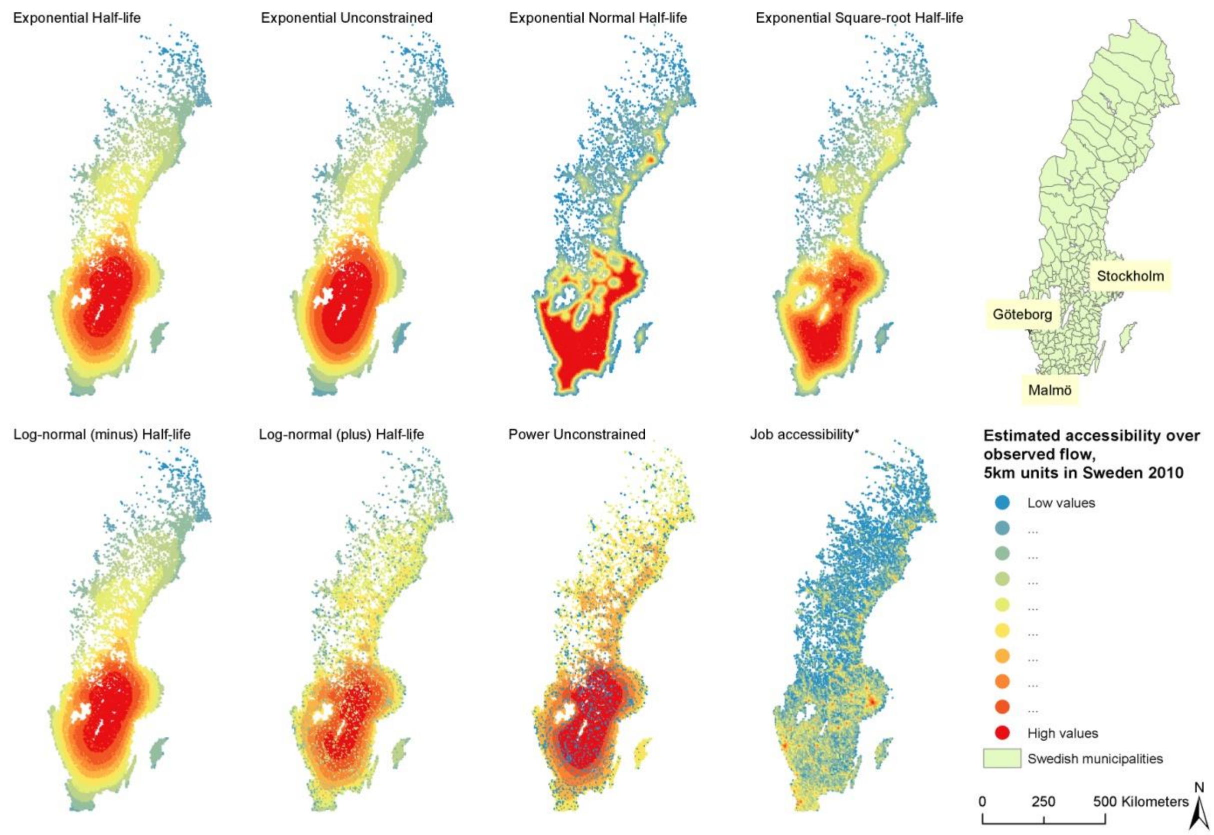

Figure 1. Illustration of normalized accessibility (potential accessibility divided by observed flow of commuters). Red colours indicate areas where the normalized values are high and blue where values are low. Maps of half-life Models (HLM) and unconstrained (UC) models show that proximity to borders, urban areas, and labour market core area in the southern parts of Sweden affects outcome differently. Upper right map shows the locations of the three major metropolitan areas in Sweden, lower right shows job accessibility (HLM exponential).

5. Conclusions

In this work, we have analysed how the distance decay parameters (which are constructed statistically), emerging from unconstrained and doubly-constrained SIMs perform in comparison with the mathematically derived parameters from HLMs, in the perspective of accessibility studies. The results reveal that doubly-constrained parameters are considerably better in datasets containing few to medium counts of units. HLMs perform similar to unconstrained models when units are few but substantially better if the count of units becomes large. In particular, doubly-constrained SIMs become increasingly difficult to compute as the number of units increase, while HLMs become more accurate (due to reduction in the systematic error between the global population median distance and unit-specific median distances).

All in all, HLMs can be considered as viable candidates for the computation of distance decay parameters especially where the count of units increase. The fact that half-life parameters can be calculated for a range of different distance decay functions means that it is reasonable to assume that they can be useful in studies of accessibility concerning short-span trips as well as long trips, such as migration. In addition, since HLMs need no statistical calibration they are easy to employ in accessibility studies, and may be employed also when observed flows between spatial units are missing, since the requested input is restricted to the median commuting distance; something that may be acquired from surveys and other alternative sources. HLMs can also be used to predict alternative accessibility scenarios by changing median distance (or time or cost) value, thereby opening up for estimation of potential accessibility under alternative settings.

The online half-life distance-decay-parameter-generator constructed for this paper can be found on this address: http://equipop.kultgeog.uu.se/Decay/untitled.html

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Appendix A: Calculation of decay parameters in HLMs

A.1. The Exponential Normal function.

As for the exponential function, the total AUC for an exponential normal function can be formulated as an integral (A1)

$$\int_0^\infty e^{-\beta x^2} dx = \frac{\sqrt{\pi}}{2\sqrt{\beta}} \tag{A1}$$

The integral for the first half of the AUC, between zero distance and median distance *m*, can be formulated as in equations (A2a and A2b):

$$\int_0^m e^{-\beta x^2} dx = \frac{0.5\sqrt{\pi}}{2\sqrt{\beta}} \tag{A2a}$$

or

$$\begin{aligned} 0.5 &= \frac{2\sqrt{\beta}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2\sqrt{\beta}} erf\left(m\sqrt{\beta}\right) \\ &= erf\left(m\sqrt{\beta}\right) \end{aligned} \tag{A2b}$$

Where $erf\left(m\sqrt{\beta}\right)$ is an error function with the argument $\left(m\sqrt{\beta}\right)$. The inverse error function version of equation A2b is expressed in equation A3.

$$\begin{aligned} erf^{-1}(0.5) &= erf^{-1}\left(erf\left(m\sqrt{\beta}\right)\right) \\ &= m\sqrt{\beta} \end{aligned} \tag{A3}$$

Solving for β yields:

$$\beta = \left(\frac{erf^{-1}(0.5)}{m}\right)^2 \tag{A4}$$

Since the inverted error function of 0.5 has the value of approximately 0.47693628, β can be expressed as:

$$\beta \approx \left(\frac{0.47693628}{m}\right)^2 \tag{A5}$$

A.2. The Exponential square-root function.

The integral for the AUC of the exponential square -root function is expressed in equation A6.

$$\int_0^\infty e^{-\beta\sqrt{x}} dx = \frac{2}{\beta^2} \tag{A6}$$

The integral for half of the AUC can be expressed as in A7a or A7b

$$\int_0^m e^{-\beta\sqrt{x}}dx = \frac{0.5*2}{\beta^2}$$
$$= \frac{1}{\beta^2} \tag{A7a}$$

or

$$\frac{1}{\beta^2} = \int_0^m e^{-\beta\sqrt{x}}dx$$
$$= \frac{2\left(1 - e^{-\beta\sqrt{m}}\left(\beta\sqrt{m+1}\right)\right)}{\beta^2} \tag{A7b}$$

Simplifying yields

$$1 = 2 - 2e^{-\beta\sqrt{m}}\left(\beta\sqrt{m+1}\right)$$
$$\leftrightarrow$$
$$-1 = -2e^{-\beta\sqrt{m}}\left(\beta\sqrt{m+1}\right)$$
$$\leftrightarrow$$
$$0.5 = e^{-\beta\sqrt{m}}\left(\beta\sqrt{m+1}\right) \tag{A8}$$

This is the same thing as solving for α in $e^{-\alpha}(\alpha+1)-0.5$ which has solution (a9)

$$\alpha = -W\left(-\frac{1}{2e}\right) - 1 \tag{A9}$$

Where $W(\bullet)$ is the Lambert W function. For arguments on the interval (-1/e,0) the function is double valued, i.e.

$$\alpha_1 \approx -0.768039$$
$$\alpha_2 \approx 1.67835 \tag{A10}$$

Since $\alpha = \beta\sqrt{m}$ the distance decay parameter β can be described either as β_1 or β_2 in equation (A11). However since only β_2 is a decaying function, this function is the preferred one for the calculation of an exponential square-root distance decay parameter.

$$\beta_1 \approx \frac{-0.768039}{\sqrt{m}}$$
$$\beta_2 \approx \frac{1.67835}{\sqrt{m}} \tag{A11}$$

A.3. The Log normal function.
The total AUC for the log normal function can be expressed as in equation (A12):

$$\int_0^{\infty} e^{-\beta(\ln x)^2} dx = \frac{\sqrt{\pi} e^{\frac{1}{4\beta}}}{\sqrt{\beta}} \qquad \text{(A12)}$$

For half of the AUC the integral is expressed as in equations (A13a and A13b)

$$\int_0^{m} e^{-\beta(\ln x)^2} dx = \frac{0.5\sqrt{\pi} e^{\frac{1}{4\beta}}}{\sqrt{\beta}} \qquad \text{(A13a)}$$

or

$$\frac{0.5\sqrt{\pi} e^{\frac{1}{4\beta}}}{\sqrt{\beta}} = \int_0^{m} e^{-\beta(\ln x)^2} dx$$

$$= \frac{\sqrt{\pi} e^{\frac{1}{4\beta}} \left(erf\left(\frac{2\beta kn(m) - 1}{2\sqrt{\beta}} \right) + 1 \right)}{\sqrt{\beta}} \qquad \text{(A13b)}$$

Simplifying the expression renders (A14):

$$-0.5 = erf\left(\frac{2\beta \ln(m) - 1}{2\sqrt{\beta}} \right) \qquad \text{(A14)}$$

And using the inverse error function:

$$erf^{-1}(0.5) = \frac{2\beta \ln(m) - 1}{2\sqrt{\beta}} \qquad \text{(A15)}$$

Solving for β can be expressed as:

$$\beta = \frac{\left(erf^{-1}(-0.5)\right)^2 \pm \sqrt{\left(erf^{-1}(-0.5)\right)^4 + 2\left(erf^{-1}(-0.5)\right)^2 \ln(m)} + \ln(m)}{2(\ln(m))^2} \qquad \text{(A16)}$$

Since the inverse error function of 0.5 has the approximate value of 0.47693628, equation (A16) can be rewritten as in equation (A17):

$$\beta \approx \frac{(0.47693628)^2 \pm \sqrt{(0.47693628)^4 + 2(0.47693628)^2 \ln(m)} + \ln(m)}{2(\ln(m))^2} \qquad \text{(A17)}$$

It is important to note that due to $\pm$ the log normal function has two alternative solutions. These solutions are in the text known as the log normal (plus) and log normal (minus) distance decay functions. This also means that in the results section both of these model variants are tested and discussed.

Appendix B: Calibration statistics

Table A1 Calibration statistics

	Municipality dataset	UC Exponential	DC Exponential	UC Power	DC Power	5km x 5km dataset	UC Exponential	DC Exponential	UC Power	DC Power
Parameters*		0.0000036	0.0000167	1.3735228	1.883558		0.0000102	*NA*	1.5285710	*NA*
R^2		0.226	0.875	0.555	0.992		0.262	*NA*	0.579	*NA*

* Table A1 Distance decay parameters used in the municipality dataset (left) and the 5km x 5km dataset (right). All decay parameters are estimated for distances measured in meters. Half-life parameters are calibrated mathematically (available in text above). Regression used for calibration of UC, Regression using iterative Newton-Raphson method used for calibration of DC.

Appendix C: Relationship between RMSE and Pearson correlation

Comparing correlation coefficients from figure A1 reveals that the best correlating variable not necessarily has the lowest RMSE value. Estimation 2 is to prefer if the exact distribution of measurement value-objects is important. If the relative distribution of value-objects is most important, estimation 1 is to prefer. Note that observed variable and estimates are purely fictitious and used in this graph only.

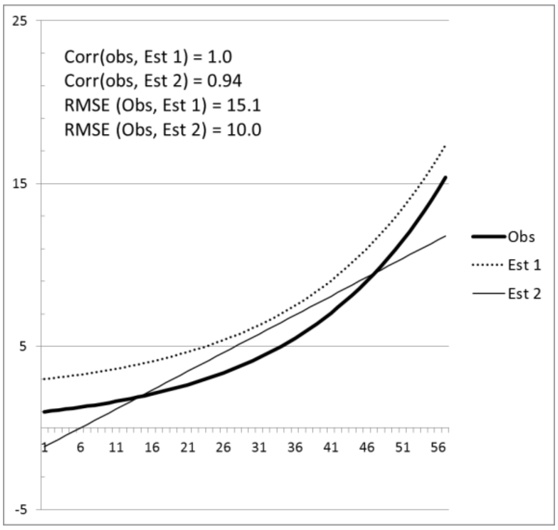

Figure A1 RMSE and Pearson correlations render results that complement each other. In this example the correlation between observed values and estimate 1 equals 1. The RMSE however reveals that the distance between observations and estimates are greater than in the alternative model where correlation reaches 0.94.

Appendix D: Half-life and Half-probability distributions

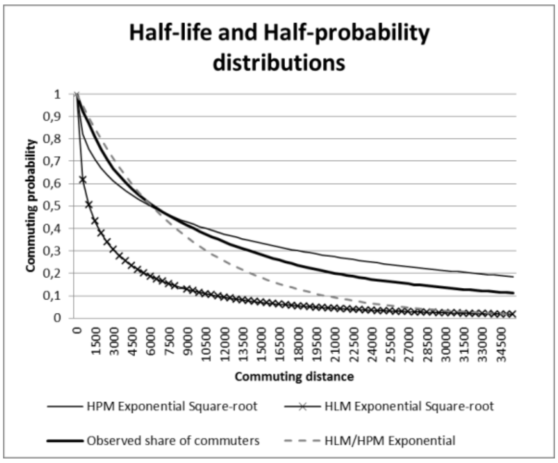

Figure A2 The median distance approach can be used differently; either the modelling approach is to split the commuting populations into two equal sized groups where one group commute shorter and the other one longer than the median distance (referred to as half-life model in the paper), or the median distance is used to determine at what distance the probability for commuting longer equals 0.5 (referred to as half-probability model or HPM). Interestingly these two approaches are united in the exponential decay function, indicating that the probability for commuting on a median distance takes place at the same distance s the two commuting population halves are equally big. For the other decay functions used in this paper, these two circumstances do not coincide. We advocate the use half-life for a reason illustrated in Figure A2 where we employ a HL-version and a HP-version of the Square-root decay function (other functions are excluded from graphics to improve visibility). If we assume that the Y-axis represents the probability of the population to commute at distances (meters on X-axis) it becomes clear that half of the commuting population, the HPM and the exponential models cross (y=0.5, x = 6010m). If we move right on the X-axis the HP-probability values remain high also over very long distances which mean that the SIMS will over-estimate interaction on longer distances. The HL-model clearly underestimates the interaction. However, since the estimated probability of commuting is highly correlated to the observed flow of commuters, the HL-model can be used to model the relative interaction.