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Modelling the effects of spatial and temporal correlation of population densities in a railway transportation corridor

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The existence of spatial and temporal correlation of population densities has been identified in many previous studies. This paper investigates the effects of spatial and temporal correlation of population densities on system disutility in a railway transportation corridor. System disutility is defined as the summation of households disutility. It shows that spatial and temporal correlation of population densities has significant effects on the results of population densities and the system performance measured in system disutility, consumer surplus and social welfare of railway system. Two numerical examples are given to illustrate the properties of the proposed model and its application together with some insightful findings.

Keywords: *Transportation, spatial and temporal correlation, population density, railway, corridor.*

1. Introduction

1.1 Motivation

In many areas, especially in cities with high population density like Shanghai and Hong Kong, households commonly make residential location choice and railway travel mode choice simultaneously (Yip et al., 2012; Li et al., 2012a; Ibeas, et al., 2013). In other words, population densities at different residential locations along an implemented railway line are correlated with each other. For instance, with given total population in a linear transportation corridor, more households choose central business district (CBD), and then less left for suburban community and new towns. In this case, negative spatial correlation exists between population densities at the CBD, and the suburban community and new towns.

Population densities in residential locations were commonly estimated with consideration of travel choices simultaneously. For instance, Lowry (1964) explained households' residential location choice on the basis of home-work journey. Recently, Waddell (2002) also determined the households residential location choice with special consideration given to the accessibility of employment.

The discrete choice models were largely used to determine the residential location choice with travel choices as the determinant factors (e.g. Anas, 1981; Eliasson and Mattsson, 2000; Pinjari et al., 2011). The model results of residential location choice were the population densities in residential locations.

The discrete choice models can help estimating population densities in residential locations, and explain the trade-offs households faced with. Nevertheless, their use has been criticized in that most of the models were proposed with an assumption of independence of irrelevant alternatives

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(IIA) hypothesis. Although this assumption was applicable in many other contexts, it was thought to be inadequate in estimation of population densities (e.g. Hunt et al., 2004).

This IIA assumption does not consider the above spatial correlation between population densities, which may significantly affect the model results of population densities. Furthermore, this assumption also ignores the temporal correlation of population densities. Population density in each residential location varies year by year. If increase of population density in the first year leads to the increase in the second year, then positive temporal correlation exists between population densities in the first year and second year, and vice versa.

To consider the spatial and temporal correlation of population densities, nested logit model (e.g. Train et al, 1987; Bhat and Guo, 2004) and C- Logit model (e.g. Cascetta et al. 1996; Zhou et al., 2012) may be alternative. Correlations between alternatives are considered in these two types of models. Train et al (1987) presented nested logit model to investigate households' choice among local telephone service options with consideration of the correlation between these choices. Bhat and Guo (2004) applied the Nest-logit model to examine residential location choice with consideration of spatial correlation of location choices. Cascetta et al. (1996) proposed C- logit model to investigate route choice with overlapping paths. Compared with Nested Logit model, C-logit model has a simple closed-form probability expression and requires relatively lower calibration effects (Zhou et al., 2012).

However, it should be noted that Nested Logit model and C-logit model only apply the correlation between alternatives to improve the estimation results in terms of choice probability of each alternative. Hence, with the application of Nested Logit model or C-logit model, the model results of population densities in residential locations are estimated values. The effects of spatial and temporal correlation of population densities cannot be examined explicitly. Optimization of population densities among residential locations cannot be conducted.

To bridge the gaps, a convex mathematical model is proposed to investigate the effects of spatial and temporal correlation on system disutility, and optimize the population densities in residential locations in this paper. The proposed model has the potential to help authorities and/or operators in dealing with many investment problems, such as the route choice of candidate railway line.

1.2 Literature review

The proposed model is essentially a sketch planning tool for railway design problems, with special consideration given to the spatial and temporal correlation of population densities.

Most previous railway design problems have aimed to minimize the system disutility. These railway design problems were optimised for large European cities which have a relatively lower population density. For instance, Vuchic and Newell (1968) proposed analytical models to optimize railway station spacing with the objective of minimizing total passenger travel time. Wirasinghe et al. (2002) developed a methodology to optimize railway terminus locations in a cross-town corridor with the objective of minimizing total passenger travel cost.

By contrast, Lam and Zhou (2000), Zhou et al. (2005), and Li et al. (2012a) suggested models for railway design problems with the objectives of profit maximization. Li et al. (2012b) presented a model to investigate the effects of integrated railway and property development on railway design with the objective of social welfare maximization. Their objectives of social welfare maximization would be appropriate to some large Asian cities with high population density, such as Shanghai and Hong Kong.

Spatial and temporal correlation of population densities are not taken into account in the above studies. This could be attributed to that railway design problems in all the above studies were considered in a single special time period. The ignorance of spatial and temporal correlation of population densities will result in inaccurate results and the ignorance of the correlation between random variables (Shao et al., 2014).

The correlation of travel demand was found to play a major role in road network expected total travel time (Waller et al, 2001). Zhao and Kockelman (2002) concluded that ignorance of correlation of travel demand would ultimately affect policy- making and infrastructure decisions. Yip et al. (2012) confirmed the existence of the correlation of travel demand by using data in Hong Kong. Travel demand between two towns and urban areas in a two 5-year period: 1996-2001 and 2001-2006 was deployed for investigation in their study.

The proposed model is essentially a sketch planning tool for railway design problems for areas no matter with low population densities or high population densities. Both system disutility and social welfare are considered in the proposed model. Since temporal correlation of population densities are taken into account, the proposed model can be used for railway design problems within a multi-period horizon.

1.3 Problem statement and contributions

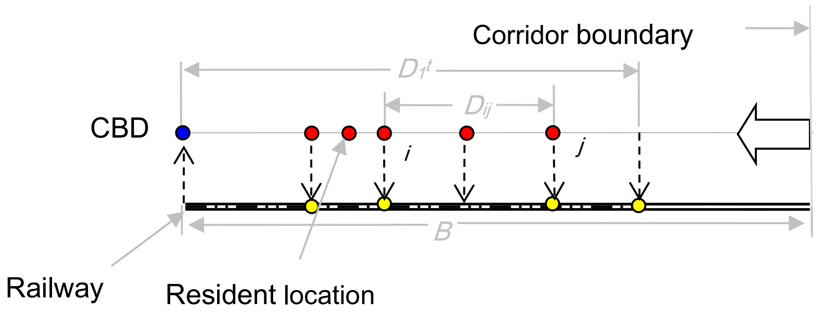

Figure 1. Configuration of residential locations in a railway transportation corridor

As shown in Figure 1, a railway transportation corridor, B km length, is proposed, which extends from the CBD towards the boundary of the corridor. The symbol D_{ij} $(\forall i, j \in [1, \mathrm{n}])$ represents the distance between residential location i and j, and D_1^t is the length of the railway line. Railway station number is denoted as n_s^t $(n_s^t \leq n)$, n is the number of residential locations in this railway transportation corridor, and n is a positive integer (Liu et al., 2009).

Population density P_i^t in each residential location i in year t is closely concerned with households' disutility U_i^t. Households' disutility U_i^t is assumed to consist of generalized travel cost π_i^t and daily housing rent r_i^t. More households' disutility U_i^t leads to less population density P_i^t, while higher population density P_i^t results in high daily housing rent r_i^t. The summation of all households' disutility U_i^t in year t is defined as system disutility U^t in year t.

The main contributions of this paper are as follows: (1) The effects of spatial and temporal correlation of population densities on system performance measured in system disutility are explored. For comparison with previous studies, its effects on consumer surplus and social welfare are also examined. (2) The population densities in residential locations are optimized under the objective of system disutility minimization.

The reminder of this paper is organized as follows. In the next section, some basic considerations are given. In Section 3, model formulation is presented. In Section 4, two numerical examples are used to illustrate the proposed model and insightful findings. Section 5 concludes this paper.

2. Basic considerations

To facilitate presentation of the essential ideas without loss of generality, some notations and assumptions are made in this paper. Notations are listed in Table 1.

Table 1. Notations

Symbol	Definition	Value
B (Km)	The length of transportation corridor	-
c (HK$)	Variable cost to supply railway service for each passenger	4
C_r (HK$/km)	Daily unit fixed maintenance cost of railway line	-
C_s (HK$)	Daily fixed operation cost of each railway station	-
D_1^t (Km)	The length of railway line in year t (t is a positive integer, and $\forall t \in [1, \mathrm{m}]$)	-
D_{ij} (Km)	The distance between residential location i and j	-
f (HK$)	Railway fare for each passenger	12
P_i^t (Persons)	Population density in residential location i in year t	-
r_i^t (HK$)	Daily housing rent in in residential location i in year t	-
U (HK$)	System disutility	-
U^t (HK$)	System disutility in year t	-
U_i^t (HK$)	Households disutility in residential location i in year t	-
π_i^t (HK$)	Generalized travel cost from residential location i to the CBD in year t	-
θ	Parameter in travel demand function of railway service	-
ξ	The cumulative probability function	-

The assumptions in the model are outlined below:

A1 All job opportunities are assumed to be supplied in the CBD. Therefore, the residential location-CBD trip is a daily compulsory activity. Households are assumed to be homogenous. They have the same preferred arrival time to the CBD. Households are assumed to make residential location choice by tradeoff of generalized travel cost and daily housing rent. (Li et al., 2012a) This assumption can be extended further. For instance, denote 80% as the proportion of trips with CBD as the destinations.

A2 The standard deviation (SD) of population density is assumed to be an increasing function with respect to its mean value. This function is referred to as the stochastic population density function in this paper. In reality, different population density levels would result in different variances of population densities. This stochastic population density function is just aimed to represent this relationship. In general case, a higher population density level would result in a larger variance of population density. Thus, the stochastic population density function is assumed as a non-decreasing function with respect to its mean value. (Lam et al, 2008)

A3 Households demand function for railway service is closely concerned with the service level. The railway service level can be captured by the generalized travel cost. Without loss of generality, an exponential demand function is used in this paper to capture the households travel demand for railway service.(Li et al., 2012b). The study period is assumed to be a one-hour period, for instance the morning peak hour, which is usually the most critical period in the day (Li et al., 2012c)

A4 Our main research objective is to investigate the effects of spatial and temporal correlation of population densities on railway system performance measured in system disutility. Thus, only the travel mode of railway is considered in this paper. This assumption is also been used in many previous studies (e.g. Wirasinghe and Ghoneim, 1981; Li et al., 2012c). This assumption can be extended into multi-modal situation, while the travel mode choice of households is examined (Li et al., 2012a).

2.1 Spatial and temporal correlation of population densities

Population density is closely related to the planning procedure of a railway line project. This data and its growth rate over years are used to make strategic decisions including: (1) whether to introduce, defer or fast-track new railway lines, (2) how long the railway should be built, and (3) how to determine effective train operation parameters, such as the number of carriages in each train, the headway between trains and the fares.

To allow for the yearly uncertainty of population density, it is assumed that there exists a perturbation in the population density. The yearly perturbed population density is given by the following equation (Yin et al., 2009):

$$P_i^t = E\left(P_i^t\right) + \varepsilon_i^t, \tag{1}$$

where P_i^t is the population density in residential location i in year t, with mean value of $E\left(P_i^t\right)$, ε_i^t is a random term, and $E\left(\varepsilon_i^t\right) = 0$. In terms of A2, the SD of population density can be expressed as (Lam et al., 2006):

$$\sigma\left(P_i^t\right) = \varphi\left(E\left(P_i^t\right)\right), \tag{2}$$

where $\sigma\left(\varphi(\cdot)\right)$ is defined as the stochastic population density function, which represents the functional relationship between the mean value and the variance of the stochastic population density.

To take spatial and temporal correlation of population density into account, the following spatial and temporal covariance is defined as:

$$\sigma_P\left(i, t_1; j, t_2\right) = \mathrm{cov}\left(P_i^{t_1}, P_j^{t_2}\right) = \rho_{i,t_1}^{j,t_2}\varphi\left(E\left(P_i^{t_1}\right)\right)\varphi\left(E\left(P_j^{t_2}\right)\right) \tag{3}$$

where $\mathrm{cov}\left(P_i^{t_1}, P_j^{t_2}\right)$ is the covariance between population density $P_i^{t_1}$ and $P_j^{t_2}$, ρ_{i,t_1}^{j,t_2} is the correlation coefficient, which is an important measurement reflecting the statistical correlation between population density $P_i^{t_1}$ and $P_j^{t_2}$. There are three correlation coefficient cases: negative, positive or zero, representing negative, positive statistical dependence or statistical independence of population density. Specifically, with $i = j$, and $t_1 = t_2$, the spatial and temporal covariance becomes the SD value.

According to the conservation law of population, we have

$$\sum_{i=1}^{n} P_i^t = P^t, \tag{4}$$

where P^t is number of the total population within the railway transportation corridor in year t.

2.2 *System disutility*

System disutility is the summation of all households disutility, namely

$$U = \sum_{t=1}^{m} \sum_{i=1}^{n} P_i^t U_i^t, \tag{5}$$

where U is system disutility, P_i^t is population density in residential location i in year t, U_i^t is households disutility in residential location i in year t.

Households disutility U_i^t is assumed to consist of generalized travel cost by railway π_i^t and daily housing rent r_i^t, namely (Lam et al., 2008)

$$U_i^t = \pi_i^t + r_i^t + \xi^{-1}(p)\sqrt{\sigma(\pi_i^t)}, \tag{6}$$

where $\xi^{-1}(p)\sqrt{\sigma(\pi_i^t)}$ is households generalized travel cost budget, $\sigma(\pi_i^t)$ is the standard deviation of generalized travel cost from residential location i to the CBD in year t, p is the required arrival probability to the CBD, $\xi(p)$ is the cumulative probability function, and $\xi^{-1}(p)$ is its inverse function. The mean value of households disutility is notated as $E(U_i^t)$ and its standard deviation is $\sigma(U_i^t)$.

In terms of Eqs. (1) and (5), system disutility U is a stochastic variable, since population density in residential location i in year t, P_i^t, is a stochastic variable. The mean value of system disutility $E(U_i^t)$ is given by

$$E(U) = E\left(\sum_{t=1}^{m} \sum_{i=1}^{n} P_i^t U_i^t\right), \tag{7}$$

where $E(U_i^t)$ is the mean value of households disutility in residential location i in year t.

The variance of system disutility $\sigma(U_i^t)$ is calculated by

$$\sigma(U) = E\left[(U - E(U))^2\right] = \sum_{t_1,t_2=1}^{m} \sum_{i,j=1}^{n} E(P_i^{t_1}) E(P_j^{t_2}) \sigma(i,t_1;j,t_2) \tag{8}$$

where $\sigma(i,t_1;j,t_2) = E\left[\left(U_i^{t_1} - E(U_i^{t_1})\right)\left(U_j^{t_2} - E(U_j^{t_2})\right)\right]$ is the spatial and temporal covariance between households utility.

2.3 *Consumer surplus and social welfare*

Consumer surplus and social welfare can be used as a performance measure of railway system. In this paper, they are considered for comparisons with the proposed model of system disutility minimization. Consumer surplus is the difference between the maximum price a consumer is willing to pay and the actual price they do pay. Households' consumer surplus of railway service in residential location i in year i CS_i^t can be calculated by

$$CS_i^t = \int_0^{q_i^t} \left(q_i^t\right)^{-1}(w)dw - q_i^t\left(\pi_i^t + \xi^{-1}(p)\sqrt{\sigma\left(\pi_i^t\right)}\right), \tag{9}$$

where $\int_0^{q_i^t} \left(q_i^t\right)^{-1}(w)dw$ is the willing to pay for railway service of households from residential location i in year t, q_i^t is the travel demand of railway service from residential location i in year t, and $\left(q_i^t\right)^{-1}$ is the inverse function of travel demand function.

In terms of A3, travel demand function for railway service from residential location i in year t, q_i^t, is assumed to be given by an exponential function shown as follows (Li et al., 2012c)

$$q_i^t = P_i^t \exp\left(-\theta\left(\pi_i^t + \xi^{-1}(p)\sqrt{\sigma\left(\pi_i^t\right)}\right)\right), \tag{10}$$

where θ is a positive constant, which responses the households sensitivity to the railway service level. The inverse function of travel demand can be obtained as follows

$$\pi_i^t + \xi^{-1}(p)\sqrt{\sigma\left(\pi_i^t\right)} = \left(q_i^t\right)^{-1}\left(q_i^t\right) = \frac{1}{\theta}\ln\frac{P_i^t}{q_i^t}. \tag{11}$$

Substitute Eq. (11) into Eq. (9), we have (Appendix A)

$$CS_i^t = \frac{q_i^t}{\theta}. \tag{12}$$

The profit of railway operator in year t, PR^t, is calculated by revenue from fare to minus the operation cost, namely

$$PR^t = \sum_{i=1}^{n_s^t} q_i^t\left(f - c\right) - D_1^t C_r - n_s^t C_s, \tag{13}$$

where f is fare for each passenger, c is the variable cost to supply railway service for each passenger, D_1^t is the length of railway line in year t, C_r is daily unit fixed maintenance cost of railway line, n_s^t is railway station number in year t, and C_s is daily fixed operation cost of each railway station.

The railway social welfare consists of the consumer surplus of households and the profit of railway operator. In terms of Eqs. (12) and (13), the railway social welfare over the years $t \in [1, m]$, RSW, is calculated by

$$RSW = 365\left(\sum_{t=1}^{m}\sum_{i=1}^{n_s^t}\left(q_i^t\left(f - c\right) + \frac{q_i^t}{\theta}\right) - D_1^t C_r - n_s^t C_s\right), \tag{14}$$

where 365 is a parameter converting daily social welfare into yearly social welfare.

3. Model formulation

To investigate the effects of spatial and temporal correlation of population densities on system disutility, a convex mathematical model with the objective of standard deviation of system disutility minimization is proposed, shown as follows:

$$\min \sigma(U) \tag{15}$$

s.t.

$$E(U) \le E_{upperbound}(U) \tag{16}$$

where $\sigma(U)$ and $E(U)$ are given by Eq. (8) and (7), respectively. This model can guarantee that the variation of system disutility is minimal with a required large enough expected value of system disutility $E_{upperbound}(U)$.

Since $\sigma(U) \ge 0$, the above mathematical program of Eqs.(15) and (16) is equivalent to the following problem

$$\min \frac{1}{2}\sigma^2(U), \tag{17}$$

s.t.

$$E(U) = E_{upperbound}(U). \tag{18}$$

Define $\alpha_i^t = \dfrac{P_i^t}{P^t}$, $\sum_{i=1}^{n} \alpha_i^t = 1$, substitute them into Eqs. (17) and (18), in terms of Eq. (8), we have

$$\min \frac{1}{2} \sum_{t_1,t_2=1}^{m} \sum_{i,j=1}^{n} \alpha_i^{t_1} \alpha_j^{t_2} \sigma(i,t_1;j,t_2), \tag{19}$$

s.t.

$$E(U) = \sum_{t=1}^{m} \sum_{i=1}^{n} \alpha_i^t E(U_i^t), \tag{20}$$

$$\sum_{i=1}^{n} \alpha_i^t = 1. \tag{21}$$

The Lagrangian function of programming Eqs. (19)-(21) is

$$L(\alpha_i^t) = \frac{1}{2} \sum_{t_1,t_2=1}^{m} \sum_{i,j=1}^{n} \alpha_i^{t_1} \alpha_j^{t_2} \sigma(i,t_1;j,t_2) - \lambda_1 \left(E(U) - \sum_{t=1}^{m} \sum_{i=1}^{n} \alpha_i^t E(U_i^t) \right) - \lambda_2 \left(m - \sum_{t=1}^{m} \sum_{i=1}^{n} \alpha_i^t \right), \tag{22}$$

Differentiating α_i^t, the Karush-Kuhn-Tucker condition is

$$\sum_{j=1}^{n} \alpha_j^{t_2} \sigma(i,t_1;j,t_2) - \lambda_1 \sum_{i=1}^{n} E(U_i^t) - \lambda_2 m = 0, \forall i \in [1,n], \tag{23}$$

$$E(U) = \sum_{t=1}^{m} \sum_{i=1}^{n} \alpha_i^t E(U_i^t), \tag{24}$$

$$\sum_{i=1}^{n} \alpha_i^t = 1. \tag{25}$$

With the value of m given, this is a system of $n+2$ variables and $n+2$ equations. Due to the strict convexity of the quadratic function Eq. (17), $\alpha^* = (\alpha_1^t, \alpha_2^t, \cdots, \alpha_n^t)$ is an optimal solution of programming Eqs. (17)-(18), if and only if it satisfies conditions Eqs. (23)-(25).

4. Numerical examples

To facilitate the presentation of the essential ideas and contributions of this paper, two illustrative examples are employed. Specifically, example 1 examines the effects of spatial and temporal correlation of population densities on system disutility, compared with traditional analysis on consumer surplus and social welfare. Example 2 presents the potential application of the proposed model on Hong Kong Western Line Extension project.

4.1 Example 1

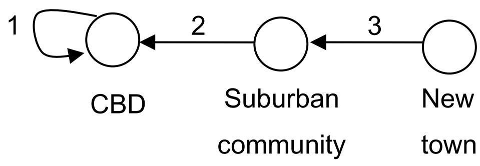

Figure 2. A toy network

For demonstration purpose, a toy network with three locations, i.e. Central Business District (CBD), suburban community and new town, is used, shown as in Figure 2. In base year, railway exists between the CBD and suburban community, and a candidate railway extension project will be implemented between suburban community and new town. Households at new town can enjoy railway service after the implementation of this extended railway line.

The Bureau of Public Roads (BPR) travel time function is employed to define the travel cost, shown as follows

$$E\left(\pi_i^t\right) = \mu t_i^0 + \gamma t_i^0 \left(0.15\left(\frac{q_i^t}{S_i^t}\right)^4\right), \sigma\left(\pi_i^t\right) = \frac{1}{5}E\left(\pi_i^t\right), \forall i = 1,2,3, \tag{26}$$

where t_i^0 is the travel time on link i by railway in year t, μ is value of time, γ is parameter for railway crowding, S_i^t is the link capacity of railway in year t. The parameters are set as $t_1^0 = 0.05$ (hour), $t_2^0 = 0.6$ (hour), $t_3^0 = 1.2$ (hour), $S_i^t = 80000$ (persons/hour), $\mu = 80$ (HK$/hour), $\gamma = 100$ (HK$/hour). The mean value of population density (persons) in each residential location in base year are given as follows (Liu et al., 2013)

$$E\left(P_1^1\right) = 5000, E\left(P_2^1\right) = 15000, E\left(P_3^1\right) = 15000, \tag{27}$$

Coefficients of variation (CV) equals to mean divided by variance. It is assumed in this example that the CV of all population densities are equal. Then, with CV=0.3, the variances of three population densities can be set as (Lam et al., 2008)

$$\sigma_P\left(1,1;1,1\right) = \left(CV \times 5000\right)^2 = \left(0.3 \times 5000\right)^2 = 1500^2 \text{ (Persons)}^2 \tag{28}$$

$$\sigma_P\left(2,1;2,1\right) = \left(CV \times 15000\right)^2 = \left(0.3 \times 15000\right)^2 = 4500^2 \text{ (Persons)}^2 \tag{29}$$

$$\sigma_P\left(3,1;3,1\right) = \left(CV \times 15000\right)^2 = \left(0.3 \times 51000\right)^2 = 4500^2 \text{ (Persons)}^2 \tag{30}$$

Housing rent function in each location is defined as (Shao et al, 2012)

$$E\left(r_i^t\right) = a_i\left(1 + b_i CMR_P\left(i,1;i,2\right)\right), \sigma\left(r_i^t\right) = \frac{1}{10}E\left(r_i^t\right), \forall i = 1,2,3, \tag{31}$$

where a_i, b_i are parameters for housing rent function, with $a_1 = 178$ (HK\$/day), $a_2 = 120$ (HK\$/day), $a_3 = 90$ (HK\$/day), and $CMR_P\left(i,1;i,2\right)$ is the covariance to mean ratio (CMR) between population density P_i^1 and P_i^2, with $CMR_P\left(i,1;i,2\right) = 0.01$. The CMR is defined as

$$CMR_P\left(i,t_1;j,t_2\right) = \frac{\sigma_P\left(i,t_1;j,t_2\right)}{\left(P_i^{t_1}P_j^{t_2}\right)^{\frac{1}{2}}}, \forall i,j \in [1,n], \forall t_1,t_2 \in [1,m]. \tag{32}$$

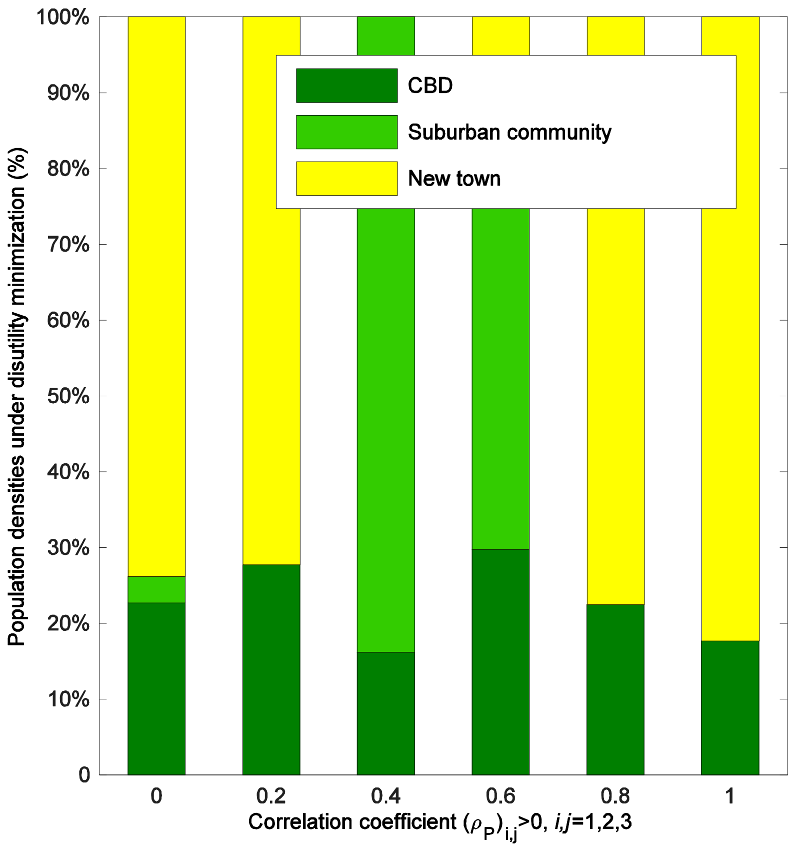

Figure 3. Population densities under disutility minimization

Figure 3 presents population densities among the CBD, suburban community and new town under system disutility minimization as spatial correlation coefficient of population densities increases from 0 to 1. In traditional studies, IIA is commonly made among population densities, namely $\left(\rho_P\right)_{i,j} = 0$ shown in Figure 3. It can be seen that underestimation of population density in the CBD exited if the actual spatial covariance of population densities was 0.2.

Table 2. Comparisons of system disutility, consumer surplus and social welfare (HK\$)

$\left(\rho_P\right)_{i,j}^t$	*System disutility*	$CS = \sum_{i=1}^{n_s^t} CS_i^t$	*RSW*
0	**7.58e+006**	0.62e+012	0.63e+012
0.2	1.13e+007	0.93e+012	1.00e+012

0.4	1.36e+007	0.31e+012	0.32e+012
0.6	1.43e+007	**1.00e+012**	1.10e+012
0.8	1.77e+007	0.61e+012	0.63e+012
1	2.08e+007	0.38e+012	0.41e+012

Table 2 shows the comparisons of the system disutility, consumer surplus and social welfare respectively, as spatial correlation coefficient $(\rho_P)_{i,j}^t$ increases from 0 to 1. The existence and uniqueness of system disutility is given in Appendix B. It can be seen that system disutility increased from 7.85e+006 HK$ to 2.08e+007 HK$ as the spatial correlation coefficient parameter $(\rho_P)_{i,j}^t$ increased from 0 to 1. In the traditional studies, $(\rho_P)_{i,j}^t = 0$, therefore, underestimation existed for system disutility in traditional studies with IIA assumption. By contrast, either underestimation or overestimation may appear for consumer surplus CS or social welfare of railway system RSW for the traditional studies with IIA assumption.

The correlation coefficient $(\rho_P)_{i,j}^t$ can be explained as the relationship between residential locations. As $(\rho_P)_{i,j}^t = 0$, it implies that all locations are not connected with each other by railway line in year t. The human communication between locations i and j does not exist in year t. In this case, system disutility is largest. As $(\rho_P)_{i,j}^t = 1$, it means that all locations adjoins each other by railway line in year t. The railway makes locations i and j inseparable. All locations become a new larger location. In this case, system disutility is not the lowest. New problem may appear for the new larger location. For instance, crowing in railway station increases the generalized travel cost.

Compared system disutility with consumer surplus and social welfare in Table 2, it can be seen that consumer surplus and social welfare is considerably larger than system disutility. The best correlation coefficient between locations is 0.6 in this example. Because when $(\rho_P)_{i,j}^t = 0.6$, the consumer surplus and social welfare is largest. Namely, locations are moderate correlated with each other. This result is approximate to correlation coefficient of Hong Kong, which is 0.4-0.5 between years 1996-2006 (Yip et al., 2012).

4.2 Example 2

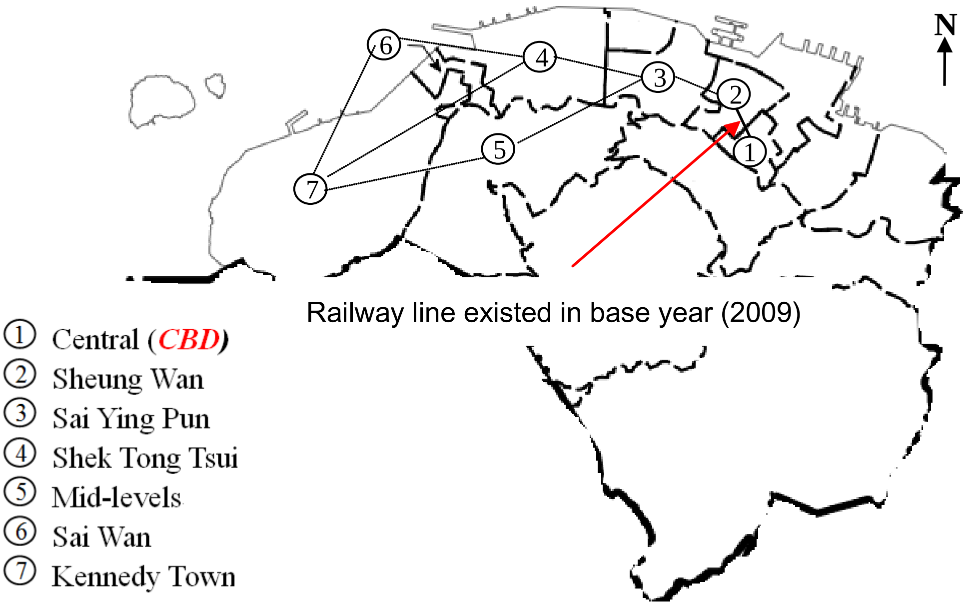

Figure 4. Residential locations in Central and Western District of Hong Kong

As shown in Figure 4, a railway extension project is planned to cater the increasing travel demand in Hong Kong Island. This extended railway line will be started from Sheung Wan denoted numbered as ◯,2. ◯,1 of Central is the Central Business District (CBD), which is connected to Sheung Wan by an existed railway line-Island line. There are other residential locations in Central and Western District of Hong Kong Island, i.e. ◯,3 Sai Ying Pun, ◯,4 Shek Tong Tsui, ◯,5 Mid-levels, ◯,6 Sai Wan, and ◯,7 Kennedy Town. Three possible railway layout schemes are available, i.e. ◯,2◯,3◯,4◯,7, ◯,2◯,3◯,5◯,7 and ◯,2◯,3◯,4◯,6◯,7, which will be compared by the proposed models in this numerical example. Set the year of 2009 as the base year here.

Preliminary

Table 3. Parameters for travel cost of the above corridor network

Link no.	○,1,○,2	○,2,○,3	○,3,○,4	○,3,○,5	○,4,○,6	○,5,○,7	○,6,○,7	○,4,○,7
D_{ij} (Km)	1.6	1.3	1.0	2.0	1.0	4.3	2.0	1.2
t_i^0 (Hour)	0.16	0.13	0.1	0.2	0.1	0.43	0.2	0.12

Table 4. Parameters for housing rent function of the above corridor network

Residential location	○,1	○,2	○,3	○,4	○,5	○,6	○,7
α_i (HK$/day)	300	300	300	250	250	250	250
b_i (HK$)	1	1	1	1	1	1	1

Travel cost functions and housing rent functions are same with example 1. The parameters of travel cost function for the above corridor network are shown in Table 3. The parameters of housing rent function for the above corridor network are shown in Table 4. These data come from Housing Department of Hong Kong Special Administrative Region.

Table 5. Mean and spatial covariance matrix of population densities in base year

	○,1	○,2	○,3	○,4	○,5	○,6	○,7	Mean	$\alpha_i^1(\%)$
○,1	624100	450300	1129700	2176450	809750	560900	2377900	7900	3.99
○,2	450300	1299600	1630200	3140700	1168500	809400	3431400	11400	5.76
○,3	1129700	1630200	8179600	7879300	2931500	2030600	8608600	28600	14.4
○,4	2176450	3140700	7879300	3036010	5647750	3912100	16585100	55100	27.8
○,5	809750	1168500	2931500	5647750	4202500	1455500	6170500	20500	10.3
○,6	560900	809400	2030600	3912100	1455500	2016400	4274200	14200	7.18
○,7	2377900	3431400	8608600	16585100	6170500	4274200	3624040	60200	30.4

Table 5 tells the mean and spatial covariance matrix of population densities in each residential location in base year of 2009. The correlation coefficient $(\rho_P)_{ij}^1$ ($\forall i, j \in [1,7]$, i, j is positive integer) is set as 0.5. These data come from Planning Department of Hong Kong Special Administrative Region.

Discussion of results

Table 6. Optimal results of population densities in each residential location

Proportion (%)	○,1	○,2	○,3	○,4	○,5	○,6	○,7
α_i^1	11.67%	17.7%	9.73%	13.83%	23.68%	22.29%	**1.1%**

Table 6 gives the optimal results of population densities in each residential location under the objective of system disutility minimization after the implementation of railway extension projects. It can be seen that the location with the highest population density proportion is Mid-levels of 23.68%, lowest is Kennedy Town of 1.1%.

From Table 6, we can know that the best railway extension layout is ○,2○,3○,4○,5○,6, not the candidate alternatives of○,2○,3○,4○,7, ○,2○,3○,5○,7 or ○,2○,3○,4○,6 ○,7, because the proportion of population densities in residential location7 is lowest, with only 1.1%.

The current population densities are only one important factor to determine where the railway station should be located. From the above example, location ○,7 is not a good choice for location choice. However, there are many other factors that authorities and/or operators should be considered, such as city planning, the construction difficulty, investment amounts, equity between different residential locations, and so on. For this railway line extension project, ○,2 ○,3 ○,5 ○,7 is the actual route which is set to open in 2014. The three railway stations are ○,3 Sai Ying Pun, Hong Kong University, and ○,7 Kennedy Town. Hong Kong University adjoins ○,5 Mid-levels.

5. Conclusions and further studies

This paper proposes a convex mathematical model to investigate the effects of spatial and temporal correlation of population densities in a transportation corridor. It shows that spatial and temporal correlation of population densities has significant effects on the population densities and the system performance measured in system disutility, consumer surplus and social welfare.

Underestimation existed for system disutility in contrast the traditional studies with IIA assumption with the proposed model. Either underestimation or overestimation may appear for consumer surplus CS or social welfare of railway system RSW for the traditional studies with IIA assumption.

It can be concluded that consumer surplus and social welfare is considerably larger than system disutility in the proposed model. The best correlation coefficient between locations is 0.6 in the proposed example 1. Because when $\left(\rho_P\right)_{i,j}^{t}=0.6$, the consumer surplus and social welfare is largest. Namely, locations are moderate correlated with each other. This results is approximate to correlation coefficient of Hong Kong, which is 04~0.5 between year 1996-2006 (Yip et al., 2012). The proposed model is also applied for analysis of Island Extension line in Hong Kong. The caveats of the proposed model are discussed as well.

Further research is needed in the following directions:

- In this paper, a railway transportation corridor is used with only one CBD and several other residential locations. The corridor boundary is not explicitly considered. It is necessary to elaborate the corridor boundary so as to extend to general network model in a further study (Li et al., 2012a).
- All households were assumed to be homogeneous and only those households commuting by railway are considered in this paper. If households travel mode behaviours are considered, more travel modes should be taken into account, such as auto, bus and park-and-ride (Liu et al., 2013). Generally, households are not homogeneous, they can be classified into many groups in the further studies (Lu et al., 2013).
- The decision for a railway design problem involves consideration of technological, social and economic factors. The prime reason could be social or in other words a desire to make life more convenient as regards manoeuvrability for a specific set of people, namely those living in the vicinity of the line and new stations to be constructed. However, only pressing economic factor is considered in this paper. Example 2 shows the caveats of this limited consideration of economic factor. More detailed social factors can be taken into account in further studies, for instance equity between different residential locations, investment amounts limitations, the construction difficulty in each residential locations.

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Appendix A

The step by step derivation of the Consumer surplus is given as follows:

$$\begin{aligned}
\mathrm{CS}_i^t &= \int_0^{q_i^t} (q_i^t)^{-1}(w)\,\mathrm{d}\,w - q_i^t(\pi_i^t + \xi^{-1}(p)\sqrt{\mathrm{var}(\pi_i^t)}) \\
&= \int_0^{q_i^t} \frac{1}{\theta}[\ln P_i^t - \ln w]\,\mathrm{d}\,w - q_i^t(\pi_i^t + \xi^{-1}(p)\sqrt{\mathrm{var}(\pi_i^t)}) \\
&= \frac{1}{\theta}[q_i^t \ln P_i^t - q_i^t \ln q_i^t + q_i^t] - q_i^t(\pi_i^t + \xi^{-1}(p)\sqrt{\mathrm{var}(\pi_i^t)}) \\
&= \frac{q_i^t}{\theta}[\ln(\frac{P_i^t}{q_i^t}) + 1] - q_i^t \frac{1}{\theta}[\ln(\frac{P_i^t}{q_i^t})] \\
&= \frac{q_i^t}{\theta}.
\end{aligned}$$

Appendix B

Theorem. System disutility $U^t\left(t=1,2,\cdots,\mathrm{m}\right)$ is calculated by population density P_i^t of each location i $\left(i=1,2,\cdots,n\right)$ multiplied with corresponding location disutility U_i^t, and feasible set of system disutility has the following properties.

- Feasible set of system disutility is a solid dimensional region.
- Feasible set of system disutility is convex to the left.

Proof. This proof will be presented in several cases: (a) two-year and two-location case and (b) over-year and multi-location case.

(a) Two-year and two-location case

Two system disutility value U^t with ($P_i^t,\overline{U}_i^t,\sigma_i^t$), where $t=1,2$ and $i=1,2$. Define $\alpha_i^t=\frac{P_i^t}{P^t}$, $\sum_{i=1}^{2}\alpha_i^t=1$, and $U=\sum_{t=1}^{m}\alpha_i^t U^t$, we have

$$U=\alpha_1^1U_1^1+(1-\alpha_1^1)U_2^1+\alpha_1^2U_1^2+(1-\alpha_1^2)U_2^2 \tag{B.1}$$

$$\overline{U}=\alpha_1^1\overline{U}_1^1+(1-\alpha_1^1)\overline{U}_2^1+\alpha_1^2\overline{U}_1^2+(1-\alpha_1^2)\overline{U}_2^2 \tag{B.2}$$

$$\begin{aligned}\mathrm{var}(U)=&(\alpha_1^1)^2(\sigma_1^1)^2+(1-\alpha_1^1)^2(\sigma_2^1)^2+(\alpha_1^2)^2(\sigma_1^2)^2+(1-\alpha_1^2)^2(\sigma_2^2)^2\\&+2\alpha_1^1(1-\alpha_1^1)\sigma_{12}^1+2\alpha_1^1\alpha_1^2\sigma_1^{12}+2\alpha_1^1(1-\alpha_1^2)\sigma_{1\to2}^{1\to2}+2(1-\alpha_1^1)\alpha_1^2\sigma_{2\to1}^{1\to2}\\&+2(1-\alpha_1^1)(1-\alpha_1^2)\sigma_2^{12}+2\alpha_1^2(1-\alpha_1^2)\sigma_{12}^2=\sigma^2(\alpha_i^t)\end{aligned} \tag{B.3}$$

Where $\sigma_{i\to j}^{u\to v}$ represents the covariance between disutility of location i in year u and that of location j in year v. Denote

$$\sigma^2(\alpha_i^t)=\psi(\overline{U}_i^t) \tag{B.4}$$

It could be found that both $\overline{U}_i^t$ and $(\overline{U}_i^t)^2$ are included in function ψ.

Define $\begin{matrix}\sigma_{12}^1=\rho_{12}^1\sigma_1^1\sigma_2^1,\sigma_1^{12}=\rho_1^{12}\sigma_1^1\sigma_1^2,\sigma_{1\to2}^{1\to2}=\rho_{1\to2}^{1\to2}\sigma_1^1\sigma_2^2,\\ \sigma_2^{12}=\rho_2^{12}\sigma_2^1\sigma_2^2,\sigma_{12}^2=\rho_{12}^2\sigma_1^2\sigma_2^2,\sigma_{2\to1}^{1\to2}=\rho_{2\to1}^{1\to2}\sigma_2^1\sigma_1^2\end{matrix}$ we have

$$\sigma(\alpha_i^t)=\sqrt{\begin{aligned}&(\alpha_1^1)^2(\sigma_1^1)^2+(1-\alpha_1^1)^2(\sigma_2^1)^2+(\alpha_1^2)^2(\sigma_1^2)^2+(1-\alpha_1^2)^2(\sigma_2^2)^2\\&+2\alpha_1^1(1-\alpha_1^1)\sigma_{12}^1+2\alpha_1^1\alpha_1^2\sigma_1^{12}+2\alpha_1^1(1-\alpha_1^2)\sigma_{1\to2}^{1\to2}+2(1-\alpha_1^1)\alpha_1^2\sigma_{2\to1}^{1\to2}\\&+2(1-\alpha_1^1)(1-\alpha_1^2)\sigma_2^{12}+2\alpha_1^2(1-\alpha_1^2)\sigma_{12}^2,\end{aligned}} \tag{B.5}$$

where $\rho_1^{12},\rho_2^{12},\rho_{12}^1,\rho_{12}^2,\rho_{1\to2}^{1\to2},\rho_{2\to1}^{1\to2}\in[-1,1]$.

Scenario (1): $\rho_1^{12}=\rho_2^{12}=\rho_{12}^1=\rho_{12}^2=\rho_{1\to2}^{1\to2}=\rho_{2\to1}^{1\to2}=1$, the upper bound is

$$\begin{aligned}\sigma^*(\alpha_i^t)&=\sqrt{[\alpha_1^1\sigma_1^1+(1-\alpha_1^1)\sigma_2^1+\alpha_1^2\sigma_1^2+(1-\alpha_1^2)\sigma_2^2]^2}\\&=\alpha_1^1\sigma_1^1+(1-\alpha_1^1)\sigma_2^1+\alpha_1^2\sigma_1^2+(1-\alpha_1^2)\sigma_2^2>0\end{aligned} \tag{B.6}$$

When α_1^1/α_1^2 varies between 0 and 1, this $\sigma^*(\alpha_i^t)$ and

$$\overline{U}=\alpha_1^1\overline{U}_1^1+(1-\alpha_1^1)\overline{U}_2^1+\alpha_1^2\overline{U}_1^2+(1-\alpha_1^2)\overline{U}_2^2 \tag{B.7}$$

define two line segments in $\overline{U}-\sigma$ space connecting $(\overline{U}_1^1,\sigma_1^1)$ and $(\overline{U}_2^1,\sigma_2^1)$, and $(\overline{U}_1^2,\sigma_1^2)$ and $(\overline{U}_2^2,\sigma_2^2)$.

Scenario (2): When $\rho_1^{12} = \rho_2^{12} = \rho_{12}^{1} = \rho_{12}^{2} = \rho_{1\to 2}^{1\to 2} = \rho_{2\to 1}^{1\to 2} = 0$,

$$\sigma^*(\alpha_i^t) = \sqrt{(\alpha_1^1)^2(\sigma_1^1)^2 + (1-\alpha_1^1)^2(\sigma_2^1)^2 + (\alpha_1^2)^2(\sigma_1^2)^2 + (1-\alpha_1^2)^2(\sigma_2^2)^2}, \tag{B.8}$$

which define two arcs in $\overline{U} - \sigma$ space on the left-handside of the above line segments corresponding to the situation when $\rho_1^{12} = \rho_2^{12} = \rho_{12}^{1} = \rho_{12}^{2} = \rho_{1\to 2}^{1\to 2} = \rho_{2\to 1}^{1\to 2} = 1$.

Scenario (3): $\rho_{12}^{1} = \rho_{12}^{2} = \rho_{1\to 2}^{1\to 2} = \rho_{2\to 1}^{1\to 2} = -1, \rho_1^{12} = \rho_2^{12} = 1$, we have

$$\begin{aligned}\sigma^*(\alpha_i^t) &= \sqrt{[\alpha_1^1\sigma_1^1 - (1-\alpha_1^1)\sigma_2^1 + \alpha_1^2\sigma_1^2 - (1-\alpha_1^2)\sigma_2^2]^2} \\ &= \left|\alpha_1^1\sigma_1^1 - (1-\alpha_1^1)\sigma_2^1 + \alpha_1^2\sigma_1^2 - (1-\alpha_1^2)\sigma_2^2\right| \\ &= \begin{cases} \alpha_1^1\sigma_1^1 - (1-\alpha_1^1)\sigma_2^1 + \alpha_1^2\sigma_1^2 - (1-\alpha_1^2)\sigma_2^2, \alpha_1^1 \geq \sigma_1^1/(\sigma_1^1+\sigma_2^1) \cap \alpha_1^2 \geq \sigma_1^2/(\sigma_1^2+\sigma_2^2) \\ \alpha_1^1\sigma_1^1 - (1-\alpha_1^1)\sigma_2^1 - \alpha_1^2\sigma_1^2 + (1-\alpha_1^2)\sigma_2^2, \alpha_1^1 \geq \sigma_1^1/(\sigma_1^1+\sigma_2^1) \cap \alpha_1^2 < \sigma_1^2/(\sigma_1^2+\sigma_2^2) \\ (1-\alpha_1^1)\sigma_2^1 - \alpha_1^1\sigma_1^1 - \alpha_1^2\sigma_1^2 + (1-\alpha_1^2)\sigma_2^2, \alpha_1^1 < \sigma_1^1/(\sigma_1^1+\sigma_2^1) \cap \alpha_1^2 < \sigma_1^2/(\sigma_1^2+\sigma_2^2) \\ (1-\alpha_1^1)\sigma_2^1 - \alpha_1^1\sigma_1^1 + \alpha_1^2\sigma_1^2 - (1-\alpha_1^2)\sigma_2^2, \alpha_1^1 < \sigma_1^1/(\sigma_1^1+\sigma_2^1) \cap \alpha_1^2 \geq \sigma_1^2/(\sigma_1^2+\sigma_2^2). \end{cases}\end{aligned} \tag{B.9}$$

When $\alpha_1^1 = \sigma_1^1/(\sigma_1^1+\sigma_2^1), \alpha_1^2 = \sigma_1^2/(\sigma_1^2+\sigma_2^2)$, $\sigma = 0$, $\sigma^1 = 0, \sigma^2 = 0$

$$\overline{U}^1 = \frac{\overline{U}_1^1\sigma_2^1 + \overline{U}_2^1\sigma_1^1}{\sigma_1^1+\sigma_2^1}, \overline{U}^2 = \frac{\overline{U}_1^2\sigma_2^2 + \overline{U}_2^2\sigma_1^2}{\sigma_1^2+\sigma_2^2}, \overline{U} = \overline{U}^1 + \overline{U}^2. \tag{B.10}$$

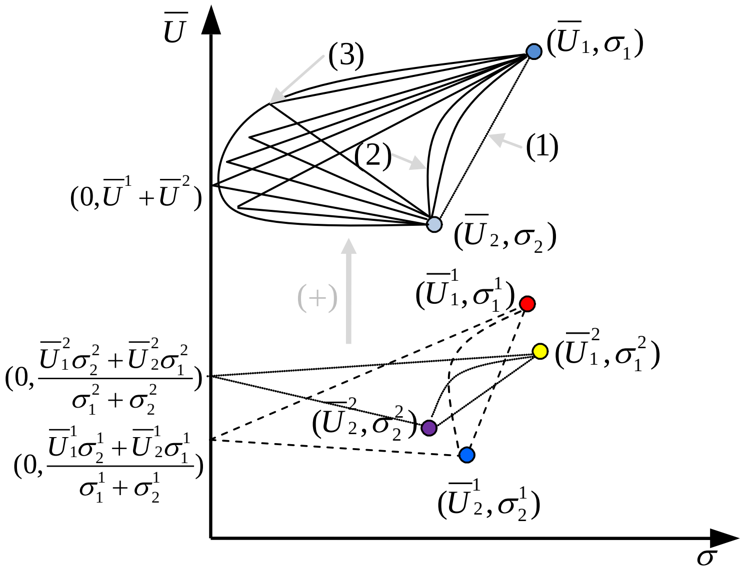

Figure 5. Feasible set of the system disutility value for two-year and two-location case

We could draw the feasible set of the total system disutility value for this two-year and two-location case as shown in additional figure shown in the above Figure 5. It is noted that scenarios (1) and (3) are normally inexistence, therefore, the feasible set area of $(\overline{U}, \sigma)$ has no point of intersection with $\overline{U}$-axis, and it lies left-handside of line segment defined with $(\overline{U}_1, \sigma_1)$ and $(\overline{U}_2, \sigma_2)$ but could not reach this line.

(b) Over-year and multi-location case

The feasible set of the over-year and multi-location total system disutility is also similar to that of additional figure. However, the area inside the feasible set curve should be biggest in all the considered cases above, which was defined by the combination of the disutility value of the considered location U_i^t in year t, where $t = 1, 2, \cdots, m$ and $i = 1, 2, \cdots, n$.

The convex feasible set of system disutility in the above theorem guarantees its existence and uniqueness.