

# Pareto-improving toll adjustment for a build-operate-transfer toll road project with unknown demand

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The social welfare gain and investment return are two main concerns of a road investment project, and they are usually key components of a Build-operate-transfer (BOT) contract. Therefore, the Pareto efficient outcomes of the two objectives are attractive for both government and private investor. This paper proposes a ex-post toll adjustment procedure called Pareto-improving trial-and-error procedure (PI-TEP), in order to achieve the Pareto efficient outcomes for a BOT toll road project with pre-determined road capacity and unknown demand function. The PI-TEP is novel and Pareto-improving, since the procedure does not require an analytical demand function and can improve social welfare and revenue simultaneously. Furthermore, the PI-TEP is theoretically proved to be efficient in obtaining the Pareto efficient outcomes. And thus, the procedure is practically useful and valuable for both government and private investor in a BOT toll road project.

**Keywords:** *toll road; trial-and-error, build-operate-transfer, pareto-improvement*

## 1. Introduction

It has long been recognized that incorporating congestion costs into road prices is essential to an efficient use of road since road use has negative congestion externality (Knight, 1924; Pigou, 1920; Walters, 1961). To internalize the congestion externality and thereby maximize social welfare, the classic marginal cost pricing principle states that road users of congestion roads should pay a toll equal to the difference between the marginal social cost and the marginal private cost (Small and Verhoef, 2007; Lindsey, 2006). A large number of studies have been conducted during the last several decades to determine and identify the optimal toll scheme in various environments (Yang and Huang, 2005). The Electronic Road Pricing system in Singapore and the central business district cordon pricing in London are two typical and successful examples of road pricing for mitigating road congestion.

Besides mitigating congestion, another purpose of road pricing is to collect toll revenue for road financing. Traditionally, large transportation infrastructure projects used to be funded primarily by governments. Therefore, there have been a number of studies looking at the linkage between

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the optimal congestion charge and road capacity investment strategy in the sense of maximizing total social welfare. In this situation, the public planner sets a user fee equal to the congestion externality. And, at the optimal capacity level, the ratio of the revenue from the congestion charge to the cost incurred for capacity equals the elasticity of construction cost with respect to capacity. The ratio is the degree of self-financing (Mohring and Harwitz, 1962; Verhoef and Mohring, 2009). The special case with constant scale economics (the elasticity is equal to one) is the exact self-financing theorem. Namely, under some conditions, the revenue from optimal congestion pricing based on the marginal cost pricing principle is sufficient to finance the fixed costs associated with the optimal capacity supply. There will be a deficit when there are economies of scale in capacity construction (the elasticity is less than one), and a surplus with diseconomies (the elasticity is larger than one). Those results have been investigated in various ways, including two-mode competition in the road (Henderson, 1985), incorporating damage of vehicles on the road (Newbery, 1988), the effect of a general transportation network (Yang and Meng, 2002) and user heterogeneity in their value of time (Arnott and Kraus, 1998).

A burgeoning literature is emerging in the road pricing area on efficiency of the private roads, motivated by global interests in road franchising (Engle et al., 1997). Apart from the development of the theory, the participation of the private firms in the road system is increasingly apparent in reality. In China, the toll roads are classified into the for-profit toll roads and the government load repayment toll roads. The highway systems in Beijing, Chongqing and Guizhou mainly consist of the for-profit toll roads. For those for-profit toll roads, the investors construct the road, operate and collect the toll revenue during their concession period (typically, 30 years) and then transfer the ownership of the roads to the government after the concession period via a build-operate-transfer (BOT) contract. The toll charge level, road capacity and duration of concession period are three key decision variables for the BOT contract between the investors and the local government, which jointly determine the social welfare gain and the return of investors. Guo and Yang (2009) and Tan et al. (2010) studied the optimal selection of the three variables for the BOT toll road contract under the assumption with deterministic traffic demand.

However, the information of traffic demand is hard to be acquired in reality. One of natural methods is to assume that the realized traffic demand is a random variable. Engle et al. (2001) proposed the least-present-value-of-revenue auction mechanism to select the optimal concessionaire. For any realized traffic demand level, the planner can obtain the optimal social welfare gain with flexible concession period. In their flexible concession-period contract, the road capacity is large enough and the road congestion is neglected, and thus, the road construction cost is pre-determined and fixed. Furthermore, the risk of the ex-post traffic demand is completely burdened by the government, which reduces the cost-cutting incentive of the private investors. To overcome the above two weaknesses, Tan and Yang (2012) proposed the partially flexible BOT contract by allowing the risk-sharing of the ex-post traffic demand and Pareto-improving ex-post toll adjustment. Niu and Zhang (2013) studied the similar problem as Tan and Yang (2012) using a stochastic bi-objective programming model.

For the BOT toll road project, the determination of the toll level is still one of the most challenge things because the exact demand function or probability distribution of the random demand level is hard obtained even after the toll road is built and operated for several years in reality. To solve the practical difficulty of demand side on decision-making and implement the socially optimal link toll scheme, Li (1999, 2002) made a notable progress in exploring the conceptual proposal and demonstrated that the exact knowledge of the demand function is not really necessary for the implementation of congestion pricing. What is required for the congestion toll estimation is that it can be derived from the engineering-based speed-flow relationship and an accurate estimation of the value of travel time savings. Li (2002) proposed an iterative procedure in deriving the "optimal" congestion toll in the absence of demand function and applied the procedure to estimate the congestion toll by using directly the commonly available traffic count data for the area-licensing scheme and major expressways in Singapore. Yang et al. (2004) extended the

iterative procedure proposed by Li (2002) and proved the convergent properties rigorously to a general network in presence of the demand function. Note that, the convergence of the trial-and-error procedure proposed by Li (2002) cannot be guaranteed by the bi-section method under certain conditions. Wang and Yang (2012) modified the procedure and proved its convergence under general conditions.

It must be pointed out that, the trial-and-error procedure (Li, 2002; Wang and Yang, 2012) is proposed to implement the social optimal toll scheme to maximize the total social welfare. However, for a toll road project with the franchising scheme, the toll revenue is an important economic index besides the social welfare. The toll revenue can be used to cover the toll road construction investment and future the operation & maintenance cost. Specially, for a BOT toll road project, a private sector builds a public road and collects the user fees to recover his/her investment returns by setting a long-term contract with the government (Nombela and de Rus, 2004; Guo and Yang, 2009, Tan et al., 2010; Tan and Yang, 2012). In this case, the toll adjustment procedure should not reduce the interests of both public and private sectors. Therefore, the trial-and-error procedure aiming to achieve the socially optimal toll level generally cannot be accepted by the private firm.

We focus our discussion on the ex post toll adjustment procedure which is Pareto-improving in both social surplus and toll revenue with predetermined road capacity and unknown analytical demand function. The toll adjustment procedure is a revised version of the trial-and-error procedure proposed by Li (2002) and called Pareto-improving trial-and-error procedure (PI-TEP). We will prove that the procedure is efficient to achieve one of Pareto optimal outcomes in the sense of the social surplus and toll revenue. And thus, the PI-TEP is self-renewal and practically useful. The rest of the paper is organized as follows. The next section first proposes the basic problem and its mathematical model. The PI-TEP is introduced and the theoretical proof of its convergence is given in Section 3. A numerical example to depict the efficiency of the procedure is described in Section 4 and conclusions together with recommendations for future research are given in Section 5.

## 2. The problem

The outcomes of the social welfare gain and investment return, related to the road users and the private investor, respectively, are mainly concerned for a private toll road project, specially, through a build-operate-transfer (BOT) contract. Generally, the concession period, the road capacity and toll charge in a BOT contract jointly determine the outcomes. However, the exact demand curve or customer's willing-to-pay function is unknown even after the highway is built. In this paper, we assume that the highway is built and the concession period and road capacity is given. The objective of the government and the private firm is to achieve the Pareto optimal outcomes in the sense of the social welfare and revenue by the ex post toll adjustment procedure without the information of the traffic demand function.

Consider a single BOT toll road project connecting two cities. Let  $B(q)$  be the willing-to-pay function or inverse demand function, which is unknown to both government and private firm, where  $q$  is the travel demand. Denote  $t(q)$  as the link travel time function. For any given toll charge  $p$ , the demand  $q$  is assumed to be determined by the following demand-supply equilibrium condition

$$B(q) = p + \beta t(q) \quad (1)$$

where  $\beta$  is the value-of-time to convert time cost unit into equivalent monetary cost unit (we consider homogeneous users only). We assume that the operation and maintenance cost is normalized to zero.

Given the toll charge,  $p$ , the unit-time (each year for example) social surplus of the highway project is the sum of the consumer's surplus and the operator's surplus, namely,

$$S(q) = CS(q) + R(q). \quad (2)$$

In eqn. (2), the traffic demand  $q$  is determined by demand-supply equilibrium (1) for given toll  $p$ ;  $CS$  is the unit-time consumers' surplus and  $R$  is the unit-time toll revenue, which can be calculated as:

$$CS(q) = \int_0^q B(w)dw - q(p + \beta t(q)) = \int_0^q B(w)dw - qB(q) \quad (3)$$

and

$$R(q) = qp = q(B(q) - \beta t(q)). \quad (4)$$

Substitute (3) and (4) into (2), the unit-time social surplus can be calculated as

$$S(q) = \int_0^q B(w)dw - \beta qt(q). \quad (5)$$

The social surplus and the toll revenue are two important outcomes for the highway project. We consider the following bi-objective problem which results in the Pareto optimal outcomes in the sense of the social surplus and toll revenue,

$$\max_{p \geq 0} \begin{pmatrix} S(q) \\ R(q) \end{pmatrix}. \quad (6)$$

The following assumption is made on  $B(q)$ ,  $t(q)$  throughout the paper.

**Assumption 1.**

- a. The inverse demand function  $B(q)$  is strictly decreasing and differentiable in  $q$ ;  $qB(q)$  is strictly concave in  $q$ .
- b. The link travel time function  $t(q)$  is a convex and increasing function of link flow  $q$ .

Since  $qB(q)$  is strictly concave and  $t(q)$  is increasing and convex in  $q$ , we know that the revenue function given by (4) is the difference between one concave and convex functions, and thus concave in demand  $q$ . In addition, from the definition of social surplus (5) and the inverse demand function  $B(q)$  is strictly decreasing, we know that  $\partial^2 S / \partial q^2 < 0$  by direct calibration. Therefore, Assumption 1 guarantees the concavities of the functions of both toll revenue and social surplus, given by (4) and (5), respectively. It must be pointed out that, Assumption 1 is not restrictive, the commonly used linear, negative exponential and power demand functions all satisfy Assumption 1 (a) and the BPR (Bureau Public Roads) link travel time function is clearly convex and increasing in  $q$ .

Additionally, we assume that  $\lim_{q \rightarrow 0} B(q) > \beta t(0)$  which means that the maximal potential benefit is larger than the free-flow travel cost of the road. The assumption follows the common sense that a newly built road surely attracts users. Under the assumption, it is clear to know that the set of the feasible solutions of the problem (6) is not empty.

For the case with deterministic and fixed demand function  $B(q)$ , Tan *et al.* (2010) investigated the Pareto optimal contract including the selection of road capacity, toll charge and concession period. When the demand function has a known structure but unknown parameters, Tan and

Yang (2012) proposed the postponement strategy to determine the optimal contract. The postponement strategy assumes the demand function can be ex-post observed and the contract adjustment follows a two-period decision procedure. Moreover, the parameter learning method, such as Bayesian learning approach, can be adopted to analyze the multi-period contract adjustment (Besbes and Zeevi, 2009). However, in many practical applications, the structure of the exact demand function is also unknown and hard to be estimated even using enough historical data and the advanced traffic demand management technologies. Therefore, the Pareto optimal outcomes are desirable, which cannot be obtained by solving the bi-objective programming problem (6) without the analytical demand function.

Many previous studies focus on how to implement the marginal-cost pricing to derive the socially optimal link flow. That is to say, they simplified the problem to achieve the polar solution of problem (6) by considering just one objective or maximizing the social surplus. In this case, the trial-and-error method proposed by Li (2002) is enough and efficient to obtain the socially optimal toll charge level. Therefore, the method is useful for a public toll road. However, for a BOT toll road project, the road is operated by the private firm through a contract with the government. The toll adjustment procedure should consider the interests of both government and private firm. And thus, the trial-and-error method aiming to obtain the socially optimal toll charge cannot be applied directly in such a toll road project. We focus our discussion on the ex post toll adjustment provision which is Pareto-improving for both social surplus and toll revenue with unknown demand function. The toll adjustment procedure is self-renewal and practically acceptable to both government and the private firm.

### 3. Pareto-improving toll adjustment provision

In this section, we first review the trial-and-error procedure proposed by Li (2002) for a single link. The traditional trial-and-error procedure tends to achieve the socially optimal toll scheme for a congestible highway without exact travel demand information. We call the procedure as socially optimal trial-and-error procedure (SO-TEP). Correspondingly, we define our trial-and-error procedure to achieve the Pareto efficient outcomes in the sense of the social surplus and toll revenue as the Pareto-improving trial-and-error procedure (PI-TEP).

The iterative SO-TEP works as follows: start with an initial targeted flow to achieve by imposing a corresponding congestion toll. We now adopt the modified bi-section method proposed by Wang and Yang (2012) to describe the trial-and-error procedure for our late reference. For the SO-TEP, the initial target domain is set to be  $[q_L^0, q_R^0]$ , where  $q_L^0 = 0$  and  $q_R^0 = q^{UE}$  is the user optimal demand level without toll. It is clear to see that  $\tilde{q} \in [q_L^0, q_R^0]$ . Suppose any initial target link flow  $q_1 \in [q_L^0, q_R^0]$ . To achieve the target link flow  $q_1$ , imposing toll charge  $p_1$  determined by

$$p = \beta q t'(q) \tag{7}$$

with  $q = q_1$ . Then the observed demand becomes  $\hat{q}_1$ , which is an aggregate result of the user's willing-to-pay. Now, the initial target domain can be updated to  $[q_L^1, q_R^1]$ , where  $q_L^1 = \max\{q_L^0, \min\{q_1, \hat{q}_1\}\}$  and  $q_R^1 = \min\{q_R^0, \max\{q_1, \hat{q}_1\}\}$ . Two conclusions can be obtained immediately from the trial-and-error iteration:  $\tilde{q} \in [q_L^1, q_R^1]$  since  $\tilde{q} \in [\min\{q_1, \hat{q}_1\}, \max\{q_1, \hat{q}_1\}]$ ; the length of the initial domain is surely reduced (Wang and Yang, 2012).

Now, let  $q_2 = (q_L^1 + q_R^1)/2$  and estimate the toll charge using eqn. (7) with  $q = q_2$ . When the new toll charge is imposed, the traffic demand on the highway is observed and denoted as  $\hat{q}_2$ . Similarly, the target domain is now updated to  $[q_L^2, q_R^2]$  using the method as described above. We now depict the SO-TEP as follows.

*The socially optimal trial-and-error procedure (SO-TEP) (Wang and Yang, 2012):*

Step 0: (initialization) Set the initial search domain  $[q_L^0, q_R^0]$  with  $q_L^0 = 0$  and  $q_R^0 = q^{UE}$ ,  $n = 1$ .

Step 1: (impose toll and observe flow) Impose a toll  $\tau_n = \beta q_n t'(q_n)$ , where  $q_n = (q_L^{n-1} + q_R^{n-1})/2$  and observe a revealed traffic flow  $\hat{q}_n$ .

Step 3: (check convergence) If  $\|\hat{q}_n - q_n\|/\|q_n\| \leq \varepsilon$ , where  $\varepsilon$  is a sufficiently small number, then stop. Otherwise, go to Step 3.

Step 4: (update target flow) Let

$$q_L^{n+1} = \max\{q_L^n, \min(q_n, \hat{q}_n)\} \quad (8)$$

$$q_R^{n+1} = \min\{q_R^n, \max(q_n, \hat{q}_n)\} \quad (9)$$

Set a new target flow  $q_{n+1} = (q_L^{n+1} + q_R^{n+1})/2$ , and  $n := n + 1$ . Go to Step 1.

Based on the SO-TEP, we obtain a series of closed nested intervals  $\Delta_n = \{[q_L^n, q_R^n]\}_{n=1}^{+\infty}$  with following properties:

$$\Delta_{n+1} \subset \Delta_n \quad (10)$$

$$|\Delta_{n+1}| \leq |\Delta_n|/2 \quad (11)$$

and

$$\tilde{q} \in \Delta_n, \forall n \geq 1. \quad (12)$$

From the nested interval theorem, we know that the series of observed demand derived from the trial-and-error procedure converges to the socially optimal level as  $n$  approaches infinity, and thus, the toll charge sequence converges to the socially optimal level. The rigorous proof can be found in Wang and Yang (2012).

As we have mentioned, the SO-TEP aims to achieve the socially optimal toll level. If the toll road is built and operated via a franchising scheme, namely, the private firm manages the toll road. Both social welfare gain and investment return are concerned. In this case, the SO-TEP is hard to be accepted by the private firm, who cares about the toll revenue after the road is built. With such a consideration, we now propose our Pareto-improving trial-and-error procedure (PI-TEP) to adjust the toll charge and guarantee the Pareto-improving outcomes in the sense of the social surplus and the toll revenue.

We first investigate the condition under which the Pareto improvement does exist. Given an original charge  $p_0$  of the toll road, the realized demand is denoted as  $q_0$ , and the corresponding congestion externality is  $\beta q_0 t'(q_0)$ . Let  $\tilde{q}$  be the socially optimal demand level at the predetermined capacity level. If the following condition is satisfied:

$$p_0 < \beta q_0 t'(q_0), \quad (13)$$

then we must have  $q_0 > \tilde{q}$ . Otherwise, if  $q_0 \leq \tilde{q}$ , we get

$$\begin{aligned} q_0 \leq \tilde{q} &\Leftrightarrow B(q_0) \geq B(\tilde{q}) \\ &\Leftrightarrow p_0 + \beta t(q_0) \geq \beta \tilde{q} t'(\tilde{q}) + \beta t(\tilde{q}) \\ &\Leftrightarrow p_0 \geq \beta \tilde{q} t'(\tilde{q}) + (\beta t(\tilde{q}) - \beta t(q_0)) \\ &\Rightarrow p_0 \geq \beta \tilde{q} t'(\tilde{q}) \geq \beta q_0 t'(q_0) \end{aligned} \quad (14)$$

which contradicts assumption (13). The last two inequalities follow from that the increasing monotonicity of the link travel time function and  $q_0 \leq \tilde{q}$ . Furthermore, under Assumption 1, both social surplus  $S(q)$  and revenue  $R(q)$  are strictly decreasing in  $q$  when  $q > \tilde{q}$  or increasing in  $p$  when  $p < \beta \tilde{q} t'(\tilde{q})$ . Condition (13) implies that if the toll is set too low at the predetermined capacity level, a proper increase in toll would improve both the social surplus and revenue.

On the other hand, if the toll is set too high, the Pareto improvement would also exist. Denote  $\bar{q}$  as the monopoly optimal demand level at the predetermined capacity level, and the corresponding monopoly toll is  $\bar{p}$ . In viewing  $p$  as the function of  $q$ , and taking the derivation of revenue  $R(q)$  in  $p$ , we know that, if  $q < \bar{q}$ , then

$$\left. \frac{\partial R(q)}{\partial p} \right|_{q=q_0} = q_0 (1 + E_{p_0}^{q_0}) < 0 \quad (15)$$

where  $E_{p_0}^{q_0} = (dq/dp)_{q=q_0, p=p_0} p_0/q_0$  is the price elasticity of demand and demand  $q$  is determined by eqn. (1) at  $p = p_0$  and  $q = q_0$ . The last inequality follows from the concavity of  $R(q)$  in  $q$  and  $q \leq \bar{q}$  since  $\bar{q}$  maximizes revenue  $R(q)$ . Condition (15) simply means that the original toll is set too high and located at the elasticity domain of demand on price or  $E_{p_0}^{q_0} < -1$ .

The following proposition shows the conditions of guaranteeing the existence of the Pareto improvement.

**Proposition 1.** *Under Assumption 1, for any given original toll charge  $p_0$ , if the realized demand  $q_0$  satisfies condition (13) or (15), then there exists a Pareto-improving trial-and-error procedure increasing both social surplus and toll revenue of the road.*

**Proof.** Let  $\bar{q}$  and  $\tilde{q}$  denote the revenue-maximizing and socially optimal demand levels, which maximize the toll revenue  $R$  and social surplus  $S$ , respectively. It is clear to see that the original demand level  $q_0 < \bar{q}$  if condition (15) holds, and  $q_0 > \tilde{q}$  if condition (13) holds since both revenue  $R$  and social surplus  $S$ , given by (4) and (5), respectively, are strictly concave. Furthermore, from the relation between the social surplus and toll revenue

$$S(q) = R(q) + \left( \int_0^q B(\omega) d\omega - qB(q) \right), \quad (16)$$

we know that

$$\frac{d(S(q))}{dq} = \frac{d(R(q))}{dq} + (-qB'(q)). \quad (17)$$

And thus, according to eqn. (17), at  $q = \bar{q}$ , we know

$$\left. \frac{d(S(q))}{dq} \right|_{q=q_0} = \left. \frac{d(R(q))}{dq} \right|_{q=q_0} + (-q_0B'(q_0)) = -q_0B'(q_0) > 0, \quad (18)$$

which implies that  $\bar{q} < \tilde{q}$ . We now know that, both revenue  $R$  and social surplus  $S$  are strictly increasing with respect to  $q$  in domain  $(0, \bar{q})$ ; both revenue  $R$  and social surplus  $S$  are strictly decreasing with respect to  $q$  in domain  $(\tilde{q}, +\infty)$ ; revenue  $R$  decreases while social surplus  $S$  increases as  $q$  increases in domain  $[\bar{q}, \tilde{q}]$ . Therefore, since condition (13) implies  $q_0 > \tilde{q}$ , we know that decreasing  $q$  or, equivalently, increasing link toll  $p$  results in the increasing of both social surplus and revenue. Similarly, since condition (15) implies  $q_0 < \bar{q}$ , we know that increasing  $q$  or, equivalently, decreasing link toll  $p$  results in the increasing of both the social welfare and revenue. The proof is completed.  $\square$

Proposition 1 depicts two cases, in which the toll adjustment can improve both social surplus and toll revenue: condition (13) holds or the original toll charge is too low, increasing toll will improve both social surplus and revenue; while condition (15) holds or the original toll charge is too high, both social surplus and revenue can be improved by decreasing the toll charge. Under the conditions described in Proposition 1, there is an incentive for both government and private firm to adjust the original toll charge.

We now move to depict our Pareto-improving trial-and-error procedure (PI-TEP). Denote  $p_0$  and  $q_0$  as the initial toll charge and the corresponding demand level. The initial unit-time toll revenue can be calculated with eqn. (4) as  $R_0 = p_0q_0$ , while the initial unit-time social surplus  $S_0$  cannot be obtained since the demand function is unknown. The PI-TEP to achieve one of the Pareto optimal outcomes is given below.

*The Pareto-improving trial-and-error procedure (PI-TEP):*

- Step 0: Set  $n = 0$ , and calibrate the initial revenue  $R_n = p_nq_n$  according to the initial toll charge  $p_n$  and corresponding demand level  $q_n$ .
- Step 1: If condition  $p_n \leq \beta q_n t'(q_n)$  holds, then set the initial searching interval  $[q_L^n, q_R^n] = [0, q_n]$  and conduct SO-TEP to achieve the socially optimal toll scheme  $\tilde{p}$ . Otherwise, go to Step 2.
- Step 2: Impose a toll  $\hat{p}_n = \beta q_n t'(q_n)$ , observe a revealed traffic flow  $\hat{q}_n$ , and calculate the current revenue  $\hat{R}_n = \hat{p}_n \hat{q}_n$ . If  $R_n \leq \hat{R}_n$ , then set the initial searching interval  $[q_L^n, q_R^n] = [0, \hat{q}_n]$  and conduct SO-TEP to achieve the socially optimal toll scheme  $\tilde{p}$  by setting the initial searching interval as  $[q_L^n, q_R^n] = [q_n, \hat{q}_n]$ . Otherwise, go to Step 3.



- Step 3: Update the link toll to  $p_{n+1} = (p_n + \hat{p}_n)/2$ , observe a revealed traffic flow  $q_{n+1}$ , and calculate the current revenue  $R_{n+1} = p_{n+1}q_{n+1}$ .
- Step 4: If  $\|p_n - p_{n+1}\| \leq \varepsilon$ , where  $\varepsilon$  is a sufficiently small number, then stop. Otherwise go to Step 5.
- Step 5: If  $R_n > R_{n+1}$ , then set  $\hat{p}_n : p_{n+1}$  and go to Step 3;
- If  $R_n \leq R_{n+1}$  and  $p_{n+1} \leq \beta q_{n+1} t'(q_{n+1})$ , then set  $[q_L^{n+1}, q_R^{n+1}] = [q_n, q_{n+1}]$  and conduct SO-TEP to achieve the socially optimal toll scheme  $\tilde{p}$ ;
- If  $R_n \leq R_{n+1}$  and  $p_{n+1} > \beta q_{n+1} t'(q_{n+1})$ , then set  $n : n+1$ ,  $p_n : p_{n+1}$ ,  $q_n : q_{n+1}$ ,  $R_n : R_{n+1}$  and go to Step 2.

The contract adjustment is inescapable after a highway project is built because of the lack of the exact demand information. The renegotiation between the government and the private firm is suggested. However, the renegotiation is expensive because of negotiation cost and the discrepancy on the ex post contract adjustment between the government and the private firm is hard to be resolved (Athias and Saussier, 2007). It is in the dilemma for the both parties between doing nothing on the ex ante contract and renegotiation (Crocker and Masten, 1991; Bajari and Tadelis, 2001). Our PI-TEP aims to achieve one of the Pareto optimal outcomes in the sense of the social surplus and toll revenue. In fact, the realized traffic demand resulted by the procedure is one of Pareto optimal solutions of problem (6). Furthermore the procedure is Pareto-improving, namely, it does not reduce the initial levels of the social surplus and toll revenue. Therefore, the adjustment can be accepted by both parties. We now move to prove the above conclusions, which are summarized into the following proposition 2.

**Proposition 2.** *The PI-TEP is Pareto-improving and the realized demand resulted by the procedure converges to one of the Pareto optimal solutions of problem (6).*

**Proof.** It is clear to see that, at any iteration  $n$ , three cases occur:  $q_n \in (0, \bar{q})$ ,  $q_n \in [\bar{q}, \tilde{q}]$  and  $q_n \in (\tilde{q}, +\infty)$ . The corresponding toll charge and revenue are denoted as  $p_n$  and  $R_n$ , respectively. According to the proof of Proposition 1, we know that inequality  $p_n \leq \beta q_n t'(q_n)$  holds is  $q_n \in (\tilde{q}, +\infty)$ . The SO-TEP achieves the socially optimal link toll, which strictly dominates the initial toll scheme  $p_n$  since both social welfare and revenue are higher at  $\tilde{p}$  than those at  $p_n$ . Now we move to consider the case with  $q_n < \tilde{q}$ , or, equivalently,  $p_n \leq \beta q_n t'(q_n)$  does not hold. In this case, updating the toll charge to  $\hat{p}_n = \beta q_n t'(q_n)$ , the realized demand  $\hat{q}_n > \tilde{q}$ . Again,  $\tilde{p}$  strictly dominates  $p_n$  when  $R_n \leq \hat{R}_n$  since  $S(\hat{q}_n) < S(\tilde{q})$  and  $\hat{R}_n = R(\hat{q}_n) < R(\tilde{q})$  with  $\hat{q}_n > \tilde{q}$ . Now, change toll charge  $\hat{p}_n$  to  $p_{n+1} = (p_n + \hat{p}_n)/2$  if  $R_n > \hat{R}_n$ , we can calculate the revenue  $R_{n+1}$  based on the observed traffic demand  $q_{n+1}$ . According to the concavities of the revenue and social surplus, we know that  $\hat{p}_n < p_{n+1} < p_n$  and  $q_n < q_{n+1} < \hat{q}_n$ . Following the similar discussion, if  $R_n \leq R_{n+1}$  and  $p_{n+1} \leq \beta q_{n+1} t'(q_{n+1})$ , then  $\tilde{p}$  strictly dominates  $p_n$ . We can conduct SO-TEP to achieve socially optimal link toll  $\tilde{p}$ . However, if  $R_n \leq R_{n+1}$  but

$p_{n+1} \leq \beta q_{n+1} t'(q_{n+1})$  does not hold, then  $p_{n+1}$  strictly dominates  $p_n$  since both revenue and social surplus are surely improved. The reason for the relationship  $S(q_{n+1}) > S(q_n)$  holding is that the social surplus is strictly increasing in domain  $(0, \tilde{q})$  and  $q_n < q_{n+1} \leq \tilde{q}$  since  $p_{n+1} > \beta q_{n+1} t'(q_{n+1})$ . Finally, if  $R_n > R_{n+1}$ , then there must exist one Pareto optimal solution of problem (6) dominating  $p_n$  in domain  $[p_n, p_{n+1}]$ . Therefore, if  $q_n$  happens to locate in domain  $q_n \in [\bar{q}, \tilde{q}]$ , then, iterating Step 3 to Step 5, the procedure stops at toll  $p_n$ ; If  $q_n$  happens to locate in domain  $q_n \in (0, \bar{q})$ , then, iterating Step 3 to Step 5 results in a strictly Pareto-improving toll scheme for the next iteration of the procedure and a Pareto optimal solution can be achieved. The proof is completed.  $\square$

Proposition 2 guarantees the convergence of the PI-TEP to a certain Pareto optimal solution of problem (6) for any initial toll level. The proof depends on the concavities of the revenue and social surplus. We now adopt the Figure 1 to graphically explain the efficiency of the PI-TEP. As shown in Figure 1, both social surplus function  $S(q)$  and revenue function  $R(q)$  are strictly concave in traffic demand  $q$ . For any initial toll charge level, the realized demand locates in one of the three domains:  $(0, q^{MR})$ ,  $[q^{MR}, q^{MS}]$  and  $(q^{MS}, +\infty)$ , where  $q^{MR}$  and  $q^{MS}$  are the demand levels that maximize the toll revenue  $R(q)$  and social surplus  $S(q)$  functions, respectively. Note that, the realized traffic demand should be up bounded in reality. Without loss of generality, we simply assume the realized traffic demand can be infinity.

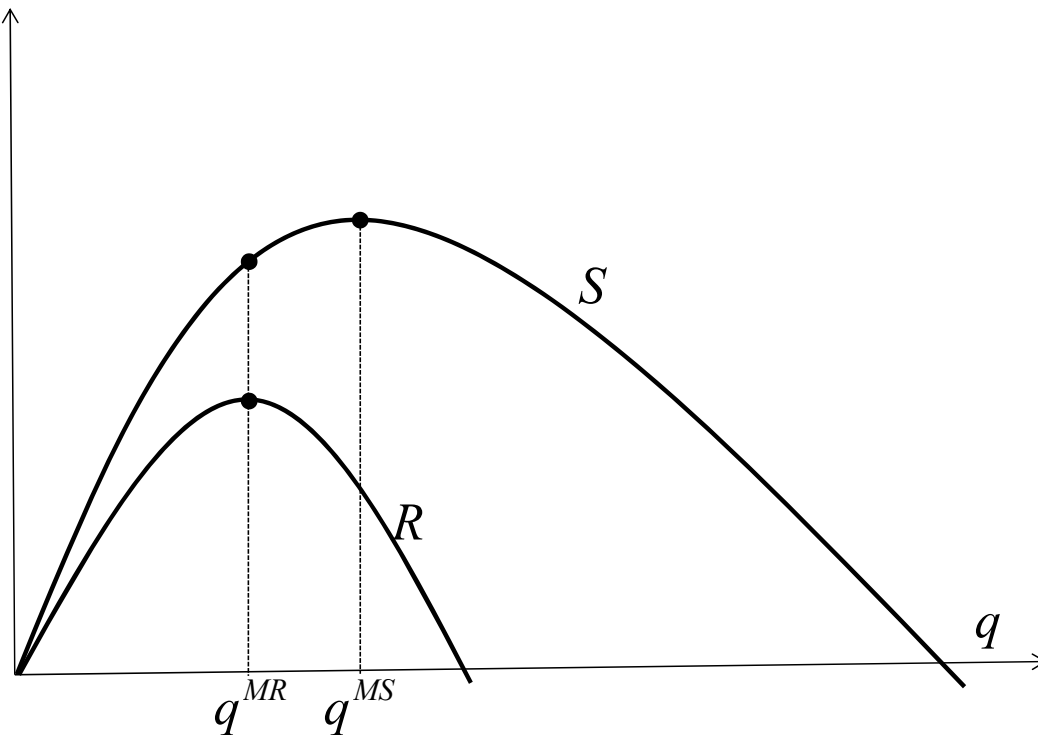


Figure 1. Graphical explanation of PI-TEP

When the realized demand  $q_0$  locates in  $(q^{MS}, +\infty)$  with the initial toll charge  $p_0$ , then condition (13) holds. In this case,  $q^{MS}$  strictly dominates  $q_0$  and is the Pareto optimal solution of problem (6). And thus, we adopt SO-TEP to implement the socially optimal toll charge.

When the realized demand  $q_0$  locates in  $[q^{MR}, q^{MS}]$  with the initial toll charge  $p_0$ , then  $q_0$  is Pareto optimal solution of problem (6). In this case, updating the toll charge using eq. (7) with  $q = q_0$ , the observed demand level must be higher than  $\tilde{q}$  and toll revenue also lower than the initial revenue level, since the revenue function is strictly decreasing in domain  $(q^{MR}, +\infty)$ , as shown in Figure 1. Using bi-section method, the PI-TEP must result in the Pareto solution  $q_0$ .

The method is a little complicate when the realized demand  $q_0$  locates in  $[0, q^{MR}]$  with the initial toll charge  $p_0$ . In this case, conducting one trial using the toll charge  $\beta q_0 t'(q_0)$ , the realized demand level must be higher than  $\tilde{q}$ . Various scenarios will occur. If the toll revenue is not lower than the initial level, then the socially optimal demand  $\tilde{q}$  must strictly dominate  $q_0$ . If the toll revenue is lower than the initial level. We can always find a demand state between  $q_0$  and new realized demand level strictly dominating  $q_0$  and repeat the procedure at the new demand state. It is clear to see that the PI-TEP must result in one Pareto optimal solution.

It must be pointed out that, condition (15) cannot be examined since the demand function is unknown. However, if the price elasticity of demand can be estimated via an external technology, then we can exactly determine the set of the Pareto optimal solutions. In this case, conditions (13) and (15) can be viewed as triggers. When either condition does not hold because of the change of the demand curve, then the PI-TEP can be triggered and the new Pareto optimal solution will be achieved. Therefore, our proposed PI-TEP is robust for the socioeconomic change. Namely, if the demand function or structure varies, one can make the procedure active to set a new adjustment scheme, which surely results in a new Pareto optimal solution based on the new demand curve.

#### 4. Numerical Example

We now present a simple numerical example to elucidate the application of the proposed Pareto-improving trial-and-error procedure (PI-TEP) for a franchising toll road project with unknown demand. The numerical example has been used in Tan et al. (2010). Consider the following BPR-type link travel time and the true demand functions

$$t(q, y) = t_0 \left( 1.0 + 0.15 \left( \frac{q}{y} \right)^4 \right) \tag{19}$$

$$B(q) = -\frac{1}{b} \ln \left( \frac{q}{Q} \right), \quad b > 0 \tag{20}$$

where  $Q$  is the potential demand and  $b$  is a scaling parameter reflecting the sensitivity of demand to full trip cost. We further assume that  $Q = 1.0 \times 10^4$  (veh/h) and  $b = 0.04$ . The free-flow travel time for the new highway  $t_0 = 0.5$  (h). The capacity of the road is set to be  $y = 1000$  (veh/h). The value-of-time is assumed to be  $\beta = 100$  (HK\$/h). The socially optimal and revenue-maximization demand levels can be calculated as  $\tilde{q} = 779$  (veh/h) and  $\bar{q} = 464$  (veh/h). The corresponding toll charges are  $\tilde{p} = 11.05$  (HK\$) and  $\bar{p} = 26.41$  (HK\$). Therefore, it is clear to see that, the Pareto optimal toll charge must locate in the domain  $[11.05, 26.41]$  and the realized demand or the set of the Pareto optimal solutions of problem (6) is  $[464, 779]$ . By simple calibration, the social surplus and toll revenue at social optimum are, respectively,  $8.61 \times 10^3$

HK\$ and  $2.81 \times 10^4$  HK\$. The social surplus and toll revenue at revenue-maximization are, respectively,  $1.23 \times 10^4$  HK\$ and  $2.39 \times 10^4$  HK\$.

We plot the toll charge, realized demand, the corresponding social surplus and toll revenue before and after the PI-TEP by varying the initial toll charge from zero to 40 HK\$ with a step 0.5. Figure 2 shows the resulted toll charge via the PI-TEP at various initial toll charge levels. It is clear to see that, if the initial toll charge is lower than the socially optimal level, then the PI-TEP derives the socially optimal toll, which is, of course, one of the Pareto optimal tolls. The PI-TEP does not change the initial toll charge which happens to be the Pareto optimal, as shown in Figure 2. When the initial toll charge is higher than the revenue-maximization level, the resulted toll charge via PI-TEP is a little random which depends on the structure of demand and travel time functions. Notably, the resulted toll charges are also Pareto optimal. In Figures 3-5, we compare the initial levels of the realized demand, social surplus and toll revenue to the ones after the PI-TEP at each given initial toll charge. The observations are similar to those in Figure 2. From Figures 2-5, it is clear to see that, our proposed PI-TEP is efficient to achieve one of the Pareto optimal outcomes in the sense of the social surplus and toll revenue with Pareto optimal toll charge and traffic demand. The procedure does not resort the exact analytical demand function.

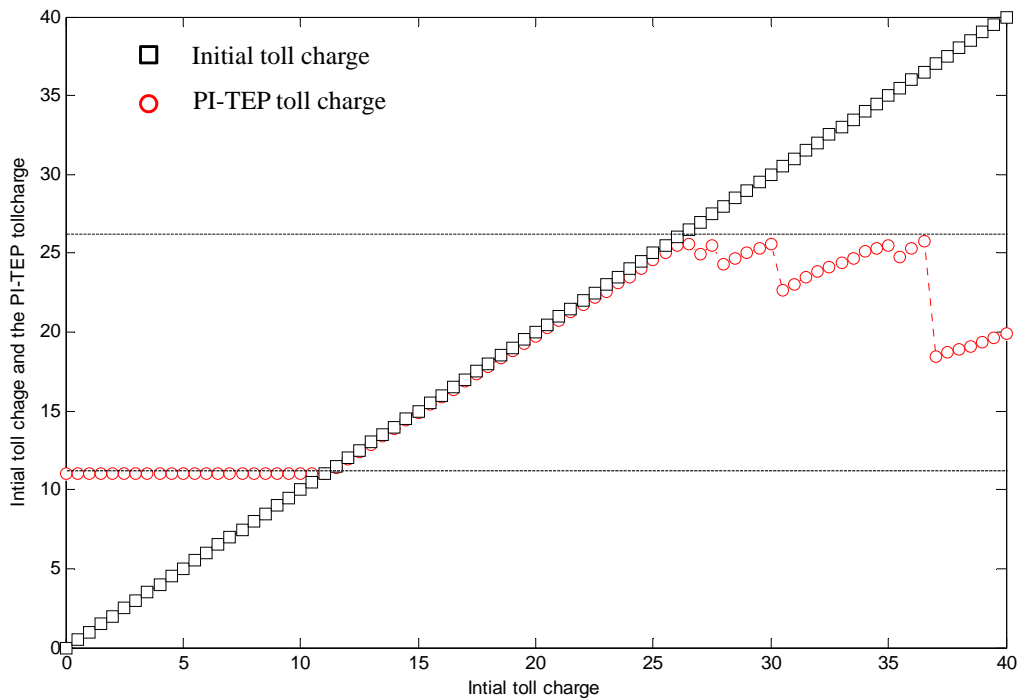


Figure 2. Toll charge before and after PI-TEP

Pareto-improving toll adjustment for a build-operate-transfer toll road project with unknown demand

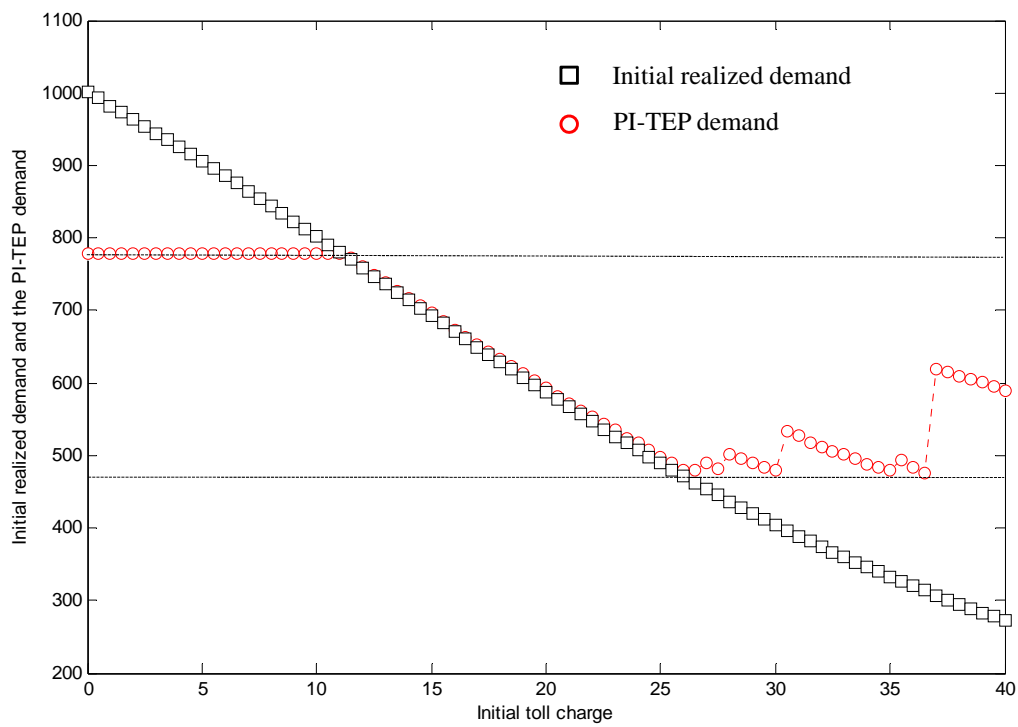


Figure 3. Realized demand before and after PI-TEP

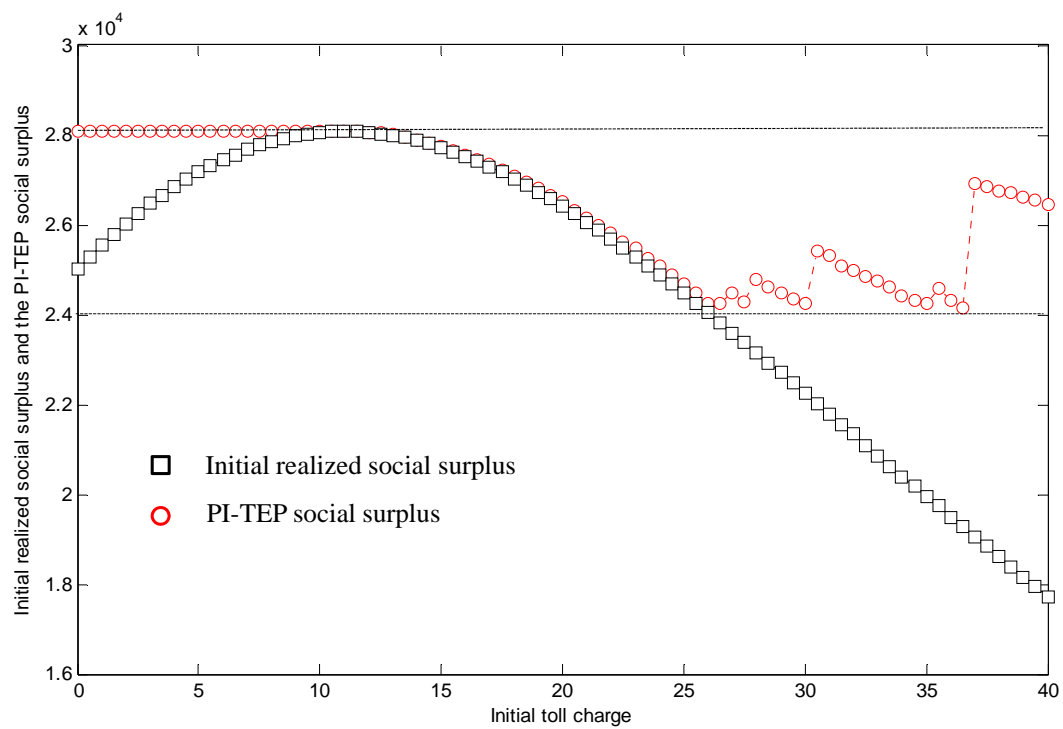


Figure 4. Realized social surplus before and after PI-TEP

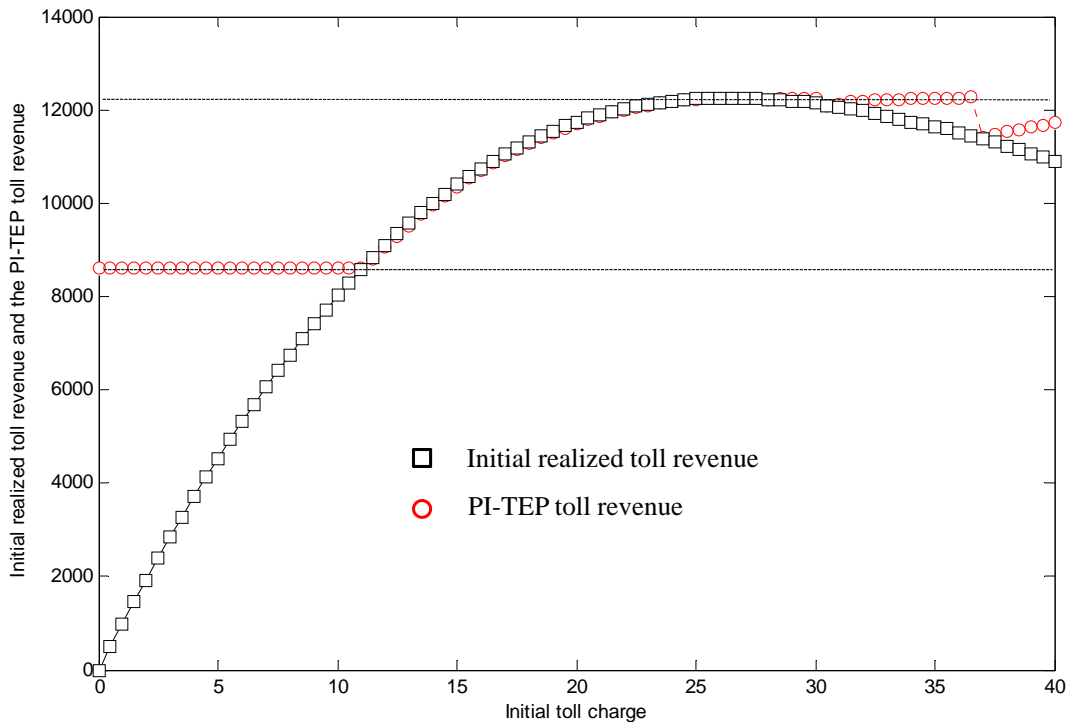


Figure 5. Realized toll revenue before and after PI-TEP

## 5. Conclusions

For a build-operate-transfer (BOT) toll road project, which is operated by the private firm, the investment return and social welfare gain are two main concerns. Therefore, the Pareto efficient outcomes in the sense of the social surplus and toll revenue are preferred by the private firm and the local government when a toll road has been built. However, the exact information on traffic demand is hard to acquire. And thus, it is impossible to ex ante select the BOT contract which is associated with a Pareto optimal outcome, as done by Tan et al. (2010). This paper proposed a Pareto-improving trial-and-error procedure (PI-TEP) to achieve the Pareto optimal outcomes with unknown demand function. Our PI-TEP is a revised version of the traditional trial-and-error procedure, which aims to derive the socially optimal toll charge. We proved that the PI-TEP is convergent and can achieve one of the Pareto optimal solutions of problem (6) under certain common assumptions. The PI-TEP is practically useful and acceptable by the private firm and government since the adjustment is Pareto-improving or it does not worsen the initial realized interest of neither party. Finally, the numerical example depicts that our PI-TEP is efficient to achieve the Pareto optimal outcomes without resorting to the analytical demand function.

It must be pointed out that the proposed PI-TEP is based on the ex post adjustment, namely, the road capacity is predetermined. One of the interesting and challenging issues is that how the ex post PI-TEP affects the ex ante decision of road capacity for a BOT toll road project. In addition, it is also interesting to improve the computing efficiency of the proposed procedure and to extend the method to a full transportation network.

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